23rd International Cosmic Ray Conference

Volume 3

CONTRIBUTED PAPERS

SH Sessions

XXIII ICRC
CALGARY 1993

July 19-30, 1993

Analytical Solution to the Transport Equation Over all the Energy Range of the Accelerated Particles.

J. Pérez-Peraza[1] and A. Gallegos-C.[2]

[1]IGUNAM-04510, C.U., [2]INAOE, Puebla-72000, MEXICO.

ABSTRACT

We present analytical particle energy spectra valid from non-to ultra-relativistic energies for stochastic Fermi acceleration. The transport equation is solved by the approximation method WKBJ. The spectra reproduces the existing numerical results and the asymptotic analytical approximations.

1. INTRODUCTION

The transport equation of accelerated particles at the source level, either in the form of a momentum diffusion equation, or, as a convection -diffusion Fokker-Planck type equation, is usually solved under certain number of simplifications: isotropie and homogeneity in space and time of the wave turbulence and isotropie of particles during their acceleration, so that their phase space density $f(p,r,t,\mu)$ becomes a pitch-angle averaged distribution $N(p,t)$. Furthermore, it is assumed time-independent particle injection into the acceleration process, and even it is often assumed energy-independent injection by considering a Delta-function. Also, it is often considered non-diffusive particle escape, by means of an energy-independent escape time, via a leaky-box loss term (τ = cst.) and an energy and time-independent acceleration efficiency value (α = cst.). Besides, it is usually assumed a thin geometry in the source, so that energy loss processes are neglected during acceleration. Under most of these simplifications the transport equation has been solved analytically by asymptotic approximations in the non-and ultra-relativistic regimes: e.g. steady-state spectra for both ranges, $E_o \ll Mc^2$ and $E_o \gg Mc^2$, have been obtained by Tverskoi 1967, Ramaty 1979, Barbosa 1979, and both, stationary and time-dependent spectra in the ultra-relativistic case have been derived e.g. by Kaplan 1956 and Melrose 1980. However, there is no analytic solution in the transrelativistic regime (Miller and Ramaty 1987; Miller et al 1990). For calculations of particle spectra over the entire energy range, numerical methods (Mullan 1980, Miller et al 1990) and Monte Carlo simulations (Miller et al 1987; Miller and Ramaty 1987) have been developed. Such methods are highly computer intensive. We present an optional method based on **WKBJ** approximation technique that allows to obtain analytical expressions for the time-dependent and equilibrium particle spectra at any energy.

2. STOCHASTIC FERMI-TYPE-ACCELERATION SPECTRA .

The well known oversimplified transport equation, describing the evolution of an accelerated particle flux $N = N(E,t)$, which is being injected at a rate $Q(E,t) = q(E)\theta(t) = q(E)$, where $\theta(t)$ is the step function, and escaping at a rate $\tau^{-1}(E,t) = \tau^{-1}$ = constant, may be written as

$$\frac{\partial N}{\partial t} = \left(\frac{1}{2}\right)\frac{\partial^2}{\partial E^2}[\langle dE^2/dt\rangle N] - \frac{\partial}{\partial E}[\langle dE/dt\rangle N] - \frac{N}{\tau} + q(E) \qquad (1)$$

where the Fokker-Planck coefficients $\langle dE/dt\rangle = A(E)$ and $\langle dE/dt\rangle = D(E)$ represent the systematic (convective) and diffusive (dispersive) energy change rates respectively: for a stochastic Fermi-like process with a momentum diffusion coefficient $D_p = (\xi/4)\alpha p^2/\beta$ and $\xi = 4/3$, $A(E) = (4/3)\alpha\beta(pc)^2 = (4/3)\alpha\beta\mathcal{E}$, $D(E) = (2/3)\alpha\beta(pc)^2 = (2/3)\alpha\beta^3\mathcal{E}^2$, where p, $\mathcal{E}$ and β are the momentun, total energy and velocity (in terms of the light speed c) of particles. The acceleration efficiency α generally depends on the paramemeters of the involved turbulence, though here is one of our free-parameters. Besides, for the goal of comparison with other authors, we consider $q(E) = q_o\delta(E - E_o)$, for $t \geq 0$.

The theory known as **WKBJ** (Wentzel, Kramer, Brillouin, Jeffrey) is a useful tool to solve differential equations of any order provided they may be linearized. The application of this method to the solution of the transport equation of accelerated particles at the level of their sources, has been previously described (Gallegos and Pérez-Peraza 1990, Pérez-Peraza and Gallegos 1993). For the specific case of stochastic Fermi-type acceleration with monoenergetic injection at E_o, we have the following results:

2.1 Steady-State Differential Spectrum

For any value $E > E_o$ the equilibrium solution of eq.(1) with continuous injection ($t \geq 0$) is

$$N(E) = \frac{(q_o/2)(\mathcal{E}/\mathcal{E}_o)^{1/2}(\beta_o/\beta)^{1/4}}{[(\alpha/3)(\overline{F} + 3/\alpha\tau)^{1/2}\beta_o^{3/2}\mathcal{E}_o]} \exp[-(\overline{F} + 3/\alpha\tau)^{1/2}\mathcal{J}] \quad \text{(part/energy)} \qquad (2)$$

where q_o(part./s) is the total number of injected particles at the energy E_o, $\mathcal{E}_o = E_o + Mc^2$, β_o is the particle velocity at E_o in terms of c, and $\mathcal{J} = \tan^{-1}\beta^{1/2} - \tan^{-1}\beta_o^{1/2} + (1/2)\ln[(1 + \beta^{1/2})(1 - \beta_o^{1/2})/(1 - \beta^{1/2})(1 + \beta_o^{1/2})]$. $\overline{F}$ is the mean value of $\overline{(dA/dE) - (d^2D/dE^2)}$, hence $\overline{F} = (\alpha/3)(\beta^{-1} + 3\beta - 2\beta^3)$ is evaluated between E_o and every energy value.

2.2 Time-Dependent Differential Spectrum.

For any energy value $E > E_o$ the time-dependent solution of eq. (1) is

$$N(E,t) = \left[\frac{4}{3}\,\pi\alpha\right]^{-1/2} (\beta_o/\beta)^{1/4} (\mathcal{E}/\mathcal{E}_o)^{1/2} (1/\beta_o^{3/2}\mathcal{E}_o)$$

$$\left\{ (N_o/t^{1/2}) \exp\left[-(\alpha t/3)\,[\bar{F} + (3/\alpha\tau)] - 3\mathcal{J}^2/4\alpha t\right] + q_o \int_o^t t'^{-1/2} \exp\left[-\,(\alpha t'/3)\,[\bar{F} + (3/\alpha\tau)] - 3\mathcal{J}^2/4\alpha t'\right] dt' \right\}$$

(part./energy) (3)

where the first term describes impulsive injection (a pulse of N_o particles at time $t = 0$) and the second one corresponds to the continuous injection at the rate q_o (part./s) at $t \geq o$.

2.3 Analytic Asymptotic Differential Spectra

Additional tests to the validity of eqs.(2)-(3), is their confrontation to analytic expressions that prove to be closed solutions of eq. (1) in a limited energy range. For Galactic Cosmic Rays and ultrarelativistic solar particles, it may be assumed that, $E_o \gg Mc^2$, within the frame of an stochastic Fermi process: the solutions of eq.(1) under the extreme assumption of $\beta \cong 1$ are:

2.3.a. Stationary Solution

$$N(E) = \frac{3q_o (E/E_o)^{-(2\,+\,3/\alpha\tau)^{1/2}}}{2\alpha E_o (2 + 3/\alpha\tau)^{1/2}} \quad \text{(part./energy)} \qquad (4)$$

2.3.b. Time-Dependent Solution

$$N(E,t) = \left[\frac{4\alpha\pi}{3}\right]^{-1/2} \left\{ (N_o/E_o)\, t^{-1/2} \exp\left[-[(2\alpha/3) + 1/\tau]t - 3\ln[E/E_o]^2/4\alpha t\right] + (q_o/E_o) \int_o^t t'^{-1/2} \left[-\,[(2\alpha/3) + 1/\tau]t' - 3[\ln(E/E_o)]^2/\,4\alpha t'\right] dt' \right\}$$

(part./energy) (5)

3. RESULTS AND CONCLUSIONS

Fig. 1. shows the equilibrium proton spectra: numerical results of Miller et al 1990 (solid line),the nonrelativistic analytical solution of Ramaty 1979 (dashed lines), the Montecarlo simula-

tion of Miller et al 1987 (crosses), and our spectrum eq.(2), (open circles).

Fig. 2 shows time-dependent proton spectra: The open circles represent continuous injection (2nd term of eq. (3). It can be appreciated that they fit quite correctly the numerical results. Fig. 3 shows the ultrarelativistic stationary spectra according to the asymptotic solution for $E_0 \gg Mc^2$ (solid lines), our asymptotic solution ($\beta \simeq 1$), eq.(4) (dashed lines) and the WKBJ spectrum valid over the entire energy range, eq.(2))(circles).It can be seen that for values of $E_0 \geq 10^5$ MeV, our spectrum eq. (2) and that of Ramaty 1979 are identical.

We conclude that the WKBJ method is a powerful tool to derive approximated analytical solutions of the Transport equation in a practically free-computer time way.

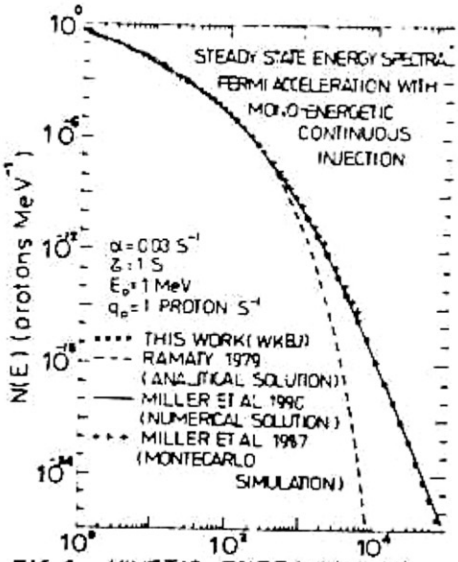

FIG.1 KINETIC ENERGY (MeV)

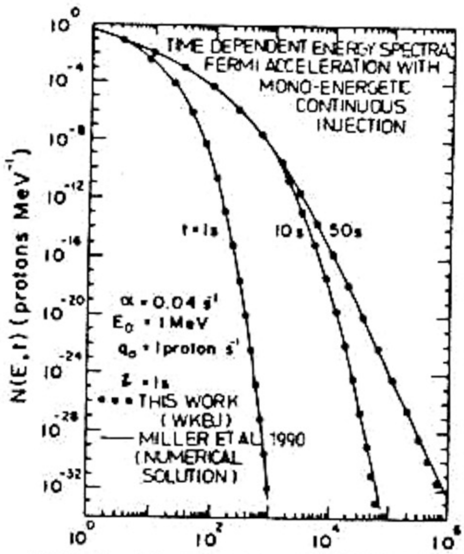

FIG. 2 KINETIC ENERGY (MeV)

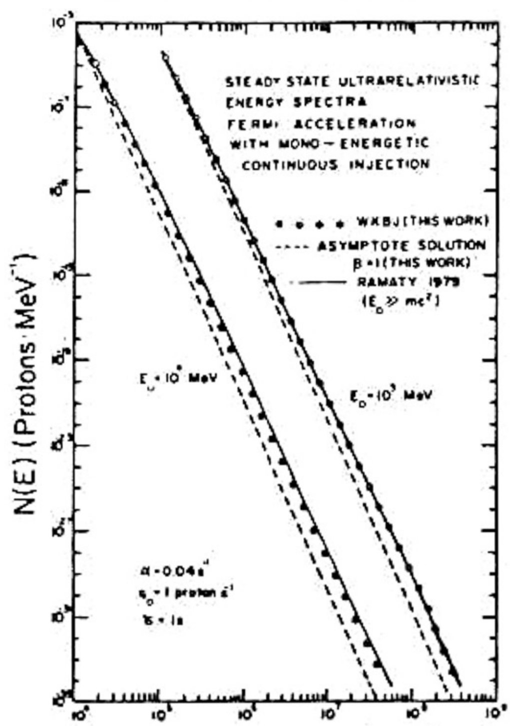

FIG.3 KINETIC ENERGY (MeV)

REFERENCES

Barbosa D.Ap.J.**233**,383,1979

Gallegos A.& Pérez-Peraza J Pro.Soc.Nuc.Mex.**1**,294,1990

Kaplan,S.A.,Soviet Phys.,**2**, 203,1956.

Melrose,D.B. 1980,in *Plasma Astrophysics*,G.& B.Publis.

Miller J. & Ramaty R.,Solar Phys.**113**,195,1987.

Miller J.,Ramaty R.& Murphy R., 20th ICRC 3,33,1987

Miller J.,Guessoum N. & Ramaty R.,Ap.J.**361**,701,1990

Mullan D.,Ap.J.**273**,244,1980

Pérez-Peraza J.and Gallegos A.,Geof.Int.**32**, 1993

Ramaty R.,*Part.Accele.Mech. in Astroph.* AIP,p.135,1979

Tverskoi B., Soviet Phys. JETP,**25**,317, 1967.