THE ASTROPHYSICAL JOURNAL, 446:400–420, 1995 June 10

DERIVATION OF ANALYTICAL PARTICLE SPECTRA FROM THE SOLUTION OF THE TRANSPORT EQUATION BY THE WKBJ METHOD

A. GALLEGOS-CRUZ
Ciencias Básicas UPIICSA, I.P.N., Té 950, Iztacalco 08400, México D.F., Mexico

AND

J. PÉREZ-PERAZA
Instituto de Geofísica, UNAM, 04510–C.U., Coyoacán, México D.F., Mexico
Received 1994 April 25; accepted 1994 December 15

ABSTRACT

We propose the use of the WKBJ method to simplify the solution of transport equations in collisionless plasmas. This technique is illustrated with the solution of a Fokker-Planck type equation, for the specific case of particle acceleration in solar flare sources. We derive analytical expressions for the time-dependent and steady state spectra that are valid over the entire energy range of the accelerated particles and fit correctly the numerical results of other authors. The derived spectra correspond to a momentum diffusion coefficient $D(p) \sim p^n/\beta$ ($n = 2$, corresponding to Fermi-type acceleration). The analytical spectrum in the transrelativistic domain is of particular importance, for instance, in the study of solar radiation emissions produced by the interaction of the energetic particles with matter and electromagnetic fields. The study is developed for two kinds of accelerating turbulence, MHD and Langmuir waves, and for monoenergetic injection, but we also present semianalytical spectra for thermal-type injection and for injection from magnetic reconnection in a neutral current sheet. We calculate the time for the time-dependent spectra to reach equilibrium at different energies, and we show that the equilibrium time is sensitive to the kind of accelerating turbulence and slightly sensitive to the kind of injection process. We analyze the effect of an energy-dependent particle escape, $\tau(E) \sim \beta^{-\mu}$ for $\mu = 0$, 1, and 2 on the particle spectrum.

Subject headings: acceleration of particles — plasmas — Sun: particle emission

1. INTRODUCTION

Transport of matter and energy is a very general problem in many fields of science, particularly in the physics of fluids. The corresponding transport equations are solved by different mathematical techniques according to the peculiarities of a particular problem. In general most of those techniques are applied when the transport equations may be reduced to a very specific kind of equation (e.g., Bessel, Legendre, etc). The method described in § 2 allows us to approach a wide range of problems since its application requires only that the transport equation be reduced to a linear differential equation. To illustrate this, in this paper we apply such a method to the specific case of determining the energy spectrum of particles during their generation in solar flares.

The knowledge of the energy distribution of energetic particles that have been accelerated in astrophysical sources is a fundamental problem within the context of phenomena taking place in plasmas. The importance is based on the fact that such distributions contain implicitly the inherent information about the properties of the acceleration process itself, the source structure and physical conditions prevailing therein during the particle generation process, and the characteristics of the traversed media during particle transport. Derivation of the energy spectra of cosmic particles may be done by several different methods, for instance, by thermodynamic equipartition between gas, particles, and electromagnetic fields (Syrovatskii 1961), or by solving the motion equation of individual particles in definite source magnetic field topologies (Pérez-Peraza, Gálvez, & Lara 1977, 1978). The formal and rigorous method is to establish a transport equation for the density of particles in the phase space: within this frame, the more general method is by means of the kinetic theory (e.g., Schlickeiser 1989), although a magnetohydrodynamic (MHD) approach is also used in the case of strong turbulence, particularly when the medium density and the magnetic field strength are high enough (e.g., Priest 1982; Byakov 1992).

Among the kinetic approaches, the most common formalisms in the study of the evolution of the energy distribution of the accelerated particles are from the Vlasov equation (collisionless Boltzmann equation); by means of the quasi-linear theory, such an equation leads to the Fokker-Planck equation for the gyrophase-averaged particle phase space density (Schlickeiser 1989). Furthermore, applying some simplifications and introducing the effects of spatial transport in a time escape (e.g., Wang & Schlickeiser 1987; Steinacker & Schlickeiser 1989), a diffusion equation in moment space is obtained. This kind of equation may be in turn transformed into a Fokker-Planck–type equation in energy space (Tystovich 1977; Melrose 1980). The latter equation can be also derived from the Chapman-Kolmogorov equation (e.g., Schatzman 1966).

In cosmic-ray physics, several terms are usually added to the previously mentioned transport equation according to the peculiarities of the scenario, as for instance, effects from the interaction of the energetic particles with matter and electromagnetic fields, source and sink effects (external injection and disappearance by escape or transformations) and so on (e.g., Ginzburg 1958; Schlickeiser 1984). In general, the establishment of the evolution equation of the energetic particles is not a difficult problem, as it is the solution of the resultant equation. The first approaches to solve analytically this type of equation were related to solutions in limited energy range: in the ultrarelativistic range Kaplan (1956), Ginzburg (1958), Kardashev (1962), Ginzburg & Syrovatskii

(1964), Tverskoi (1967), Ramaty (1979), and Melrose (1980); in the nonrelativistic range Tverskoi (1967), Ramaty (1979), and Barbosa (1979). Analytical steady state solutions over the entire momentum space have been obtained by Schlickeiser (1984), Dröge & Schlickeiser (1986), and Steinacker & Schlickeiser (1989). Time-dependent solutions have been obtained numerically by Mullan (1980) and Miller, Ramaty, & Murphy (1987) in the nonrelativistic range and by Miller, Guessoum, & Ramaty (1990) through the entire energetic range.

In § 2 we apply the WKBJ method to solve transport equations in the volume of particle acceleration to obtain time-dependent and steady state solutions over the entire energy range of the energetic particles. We illustrate the method for the case in which the power of the particle momentum in the momentum diffusion coefficient is $n = 2$. In § 3 we discuss the rates and efficiencies of acceleration of MHD and Langmuir turbulence. The injection spectra into the acceleration process is discussed in § 4. Results are presented in § 5, where we give the analytical energy spectra for both types of turbulence, assuming monoenergetic injection, and semianalytical spectra by assuming thermal-type injection and injection from a magnetic neutral current sheet (MNCS). The comparison of our results with the results that may be obtained in limiting energy ranges with closed solutions and mumerical results is also presented in § 5. We present results relative to the equilibrium time for a time-dependent spectrum to reach a steady state at a given energy, and results regarding an energy dependent escape time. The accuracy of the WKBJ method is discussed in § 6, where we suggest the use of such method as an accessible form to derive particle energy spectra in broad energy ranges.

2. SOLUTION OF THE TRANSPORT EQUATION BY THE WKBJ METHOD

Let us assume that particles are accelerated by a stochastic process, such as those derived from relaxation of MHD and Langmuir turbulence, etc. Within this context, particles gain energy through interactions with turbulence, undergoing small energy changes, so that interactions may be seen as independent events; this allows one to study the evolution of particle fluxes from the statistical point of view, by means of the distribution function of the interacting particles and the establishment of the corresponding transport equation. Within the kinetic scheme of the Vlasov equation, one assumes a noncollisional behavior of the accelerated particles among themselves, such that only interactions with the accelerating agents take place, and eventually with matter and electromagnetic fields in the medium. In a wider context, within the kinetic schema, a more general equation is often employed that contains not only stochastic acceleration by turbulence (of the type of Fermi second order), but also Fermi first-order–type acceleration, including spatial transport (e.g., Schlickeiser 1989 and references therein): they assume weak electromagnetic field turbulence of small amplitude, homogeneous in space and time in a given reference frame (e.g., Schlickeiser 1994). Under those conditions, the gyrophase-averaged particle phase space density evolves according to the Fokker-Planck equation (e.g., Schlickeiser 1989), such that in the particular case of low-frequency MHD turbulence (whose magnetic field components $\delta B \gg \delta E$-electric field components) and particle density highly isotropic (that which can be produced by an external agent, e.g., Atcherberg 1981; Melrose 1986) the Fokker-Planck equation can be reduced to a diffusion-convection equation for the pitch angle–averaged part of the phase space density (Jokipii 1966; Haselmann & Wibberentz 1968; Forman & Webb 1985; Schlickeiser 1989) which in its most general form contains terms of spatial transport (diffusion and convection) and terms of acceleration (momentum convection and momentum diffusion). The scattering time method has been used to separate the spatial transport from the momentum transport part of the equation by an approximated procedure that may be applied under certain circumstances (Wang & Schlickeiser 1987; Steinacker & Schlickeiser 1989) where spatial convection and diffusion together with catastrophic losses are combined in the escape time: $f(r, p, t)$ is then expressed as an infinite sum of momentum-dependent distribution functions $F_n(p, t)$, in such a form that second-order and higher order terms are neglected because the main contribution is given by the first term. Such a neglect is widely justified for momentum values of the solution far away from the injection threshold value p_0; however, as in any solution of a diffusion equation these higher order terms become highly important at the level of the sources (around p_0). Furthermore, when it is considered only acceleration by small amplitude MHD turbulence (second-order Fermi type) the diffusion-convection equation in momentum space for the pitch angle–averaged particle density is reduced to an equation that contains a term of escape, and eventually a source function (e.g., Dröge & Schlickeiser 1986). Some less general formalisms of the problem which do not consider spatial transport in the establishment of the equation also lead to the following well-known momentum-diffusion equation (e.g., Rosenbluth, MacDonald, & Judd 1957; Tsytovich 1977; Melrose 1980):

$$\frac{\partial f(p, t)}{\partial t} = \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2 D(p)\frac{\partial f(p, t)}{\partial p}\right]. \tag{1}$$

Obviously, this is a particular case of the transport equation as derived by Dröge & Schlickeiser (1986). Here $f(p, t)$ is the pitch angle averaged–density of particles of momentum p interacting with turbulence at time t, and $D(p)$ is the diffusion coefficient characterizing the interaction dynamics between particles and the specific type of turbulence, which is assumed to be homogeneous and time independent (Tsytovich 1977; Melrose 1980). For Landau-Cherenkov resonance $D(p) \sim p^n/\beta$, and for gyroresonance $D(p) \sim \mathscr{R}^n/\beta$, (where $\mathscr{R} = [c/Ze]p$ is the particle magnetic rigidity), we can write $D(p) = D^* p^n/\beta$, where D^* contains the information about the specific acceleration efficiency and the kind of particle. Owing to the presence of the local magnetic and photonic fields and matter, the acceleration process is not an isolated process, but particles may actually simultaneously undergo other processes that continuously modify their momentum distribution $f(p, t)$ as for instance energy losses from synchrotron radiation, inverse Compton–scattering, bremsstrahlung, Coulomb collisional interactions (nuclear and electronic stopping), adiabatic energy changes, and energy degradation by nuclear collisions, as well as other source effects such as external injection, or sink effects such as nuclear transformation or other catastrophic losses, escape by spatial diffusion, drifts, convection, etc. Sink effects are usually considered by a characteristic escape time, τ (e.g., Ginzburg 1958), and are sometimes parameterized by a momentum-dependent escape time (e.g., Lerche & Schlickeiser 1985). These additional effects may be incorporated into the particle evolution equation (1) to obtain a more realistic description of cosmic particle spectra (e.g., Schlickeiser 1984; Dröge & Schlickeiser 1986). Ramaty (1979) has solved equation (1) in energy space in the time-dependent and steady state cases for the limiting ranges of nonrelativistic and ultrarelativistic energies, with either no escape or an energy independent escape time τ. Besides, by combining first- and second-order Fermi acceleration and shock wave–regulated escape with diffusion and convection, Steinacker & Schlickeiser (1989) derived solar particle spectra from the steady state solution of the resultant momentum diffusion equation, which holds for the entire particle momentum range; however, the time-dependent solution of the composed momentum-diffusion equation has not yet derived, nor has an analytical solution over the entire momentum space from nonrelativistic to ultrarelativistic velocities been obtained.

An alternative route to solve equation (1) is by its transformation into a Fokker-Planck–type equation in the energy space of particles (e.g., Ginzburg & Syrovatskii 1964; Tsytovich 1977; Ramaty 1979; Melrose 1980):

$$\frac{\partial N(E, t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial E^2}[D(E)N(E, t)] - \frac{\partial}{\partial E}[B(E)N(E, t)] , \quad (2)$$

where E is the particle kinetic energy, and $N(E, t) = 4\pi p^2 f(p, t)/v$ is the number of particles per energy interval at time t, $D(E) = \langle dE^2/dt \rangle = 2v^2 D(p)$ is the diffusive energy change rate produced by the dispersion in energy gain around the value of the systematic energy gain rate, given by $B(E) = \langle dE/dt \rangle = p^{-2}(\partial/\partial p)[p^2 v D(p)]$. By arguments similar to those of Ginzburg (1958) or Schlickeiser (1984), the effect of systematic energy losses or any other systematic acceleration effect may be introduced in the second term of the right-hand of equation (2) by setting $A(E) = B(E) \pm$ additional systematic energy change processes. Also, a source term $Q(E, t)$ is added, (indicating external particle injection into the acceleration region) and a sink term; that, in the first instance, may be assumed to describe any kind of particle disappearance process from the acceleration volume by means of characteristic disappearance time $\tau(E, t)$. Thus, employing these arguments, equation (2) is usually rewritten as

$$\frac{\partial N(E, t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial E^2}[D(E)N(E, t)] - \frac{\partial}{\partial E}[A(E)N(E, t)] - \frac{N(E, t)}{\tau(E, t)} + Q(E, t) . \quad (3)$$

Here, $A(E)$ contains the systematic effect of stochastic acceleration and deceleration processes as well as any eventual secular (deterministic) energy change effect. $D(E)$ contains the diffusive effects due to dispersion around the systematic energy change rate $A(E)$. Although to solve equation (3) a number of simplifications are usually performed, there is not at present an analytical time-dependent solution for the entire particle energy range. Analytical expressions have been derived only in the asymptotic ranges, $E \ll mc^2$ and $E \gg mc^2$ (mc^2 = particle rest mass) (e.g., Melrose 1976, 1980; Barbosa 1979; Ramaty 1979). Particularly, the time-dependent solution in the transrelativistic range has been studied only by Monte Carlo simulations or numerical methods (e.g., Miller et al. 1987, 1990), but such methods are highly computer intensive. However, as pointed out by Miller & Ramaty (1987) and Miller et al. (1990) the spectrum of protons in the transrelativistic region is very important for the production of neutrons, pions, and gamma-nuclear lines in solar flares. Nevertheless, in the steady state situation the transrelativistic range can be studied by the analytical solution of Steinacker & Schlickeiser (1989).

Among the usual simplifications taken to solve equation (3) are to assume time independence for the escape and injection functions as well as a time-independent and energy-independent acceleration efficiency (constant). Also, a nondiffusive particle escape is considered, by means of an energy-independent escape time, via a leaky-box loss term (τ = constant) or occasionally via $\tau = \tau(\beta)$ (e.g., Dröge & Schlickeiser 1986). In addition, a thin geometry is often considered for the source so that energy loss processes are neglected during acceleration, what is valid rather in flares of long-duration. To avoid some of these simplifications we herein propose the use of the WKBJ technique to solve equation (3) over the complete energy range of the accelerated particles. The WKBJ technique is a useful tool with which to solve linear differential equations of any order. The application of this method to the solution of the transport equation of accelerated particles at the levels of their sources has been preliminarily reported by Gallegos & Pérez-Peraza (1990) and Pérez-Peraza & Gallegos (1994a). Herein, we will describe it more extensively. Since we have no confident inferences about the time dependence of the injection process, we retain, for simplicity, the general assumption that the flux $N(E, t)$ is being injected at a rate $Q(E) = q(E)\Theta(t) \cong q(E)$ [where $\Theta(t)$ is the step function] and is escaping at a rate τ^{-1}.

2.1. *Solutions for Any D(p)*

2.1.1. *Time-dependent Solution for Any D(p)*

Equation (3) cannot be exactly solved, unless $A(E)$ and $D(E)$ have a very peculiar functional form; in the case of the Fermi acceleration process, this occurs when the particle velocity in terms of the light velocity, $\beta = 1$. The exact solution for this case is derived in Appendix B. For a solution with any β value some approximations must be done, provided that no substantial information is lost by applying a given kind of approximation.

For the time-dependent solution we propose the following approach; we will later test the validity of such approximation. Let us take the Laplace transform of equation (3), which reduces to

$$s\tilde{N}(E, s) - N(E, 0) + \frac{d}{dE}[A\tilde{N}(E, s)] - \frac{d^2}{dE^2}[D\tilde{N}(E, s)] + \frac{\tilde{N}(E, s)}{\tau(E)} = \bar{Q}(E, s) ,$$

where $A = A(E)$, $D = D(E)$, and s is the Laplace variable. Developing the third and fourth term of the previous equation and reordering them yields

$$\frac{d^2\tilde{N}}{dE^2} - \left[\frac{A}{D} - \frac{2}{D}\left(\frac{dD}{dE}\right)\right]\frac{d\tilde{N}}{dE} - \left[s + \frac{1}{\tau} + \frac{dA}{dE} - \frac{d^2D}{dE^2}\right]\frac{\tilde{N}}{D} = -\frac{\bar{Q}(E, s)}{D} - \frac{N(E, 0)}{D} .$$

Now, defining

$$P_1 = -[(A/D) - (2/D)(dD/dE)] ,$$
$$P_2 = -[s + (1/\tau) + (dA/dE) - (d^2D/dE^2)]/D = -[s + a(E)]/D , \qquad (4)$$

and

$$P_3 = -[\bar{Q}(E, s) + N(E, 0)]/D ,$$

equation (3) becomes

$$\frac{d^2\tilde{N}(E, s)}{dE^2} + P_1 \frac{d\tilde{N}(E, s)}{dE} + P_2 \tilde{N}(E, s) = P_3(E, s) , \qquad (5)$$

where $\tilde{N}(E, s)$ denotes the Laplace transform of $N(E, t)$.

By first making the variable change $d\eta = g^{-1}dE$ with $g = \exp[\int_{E_0}^{E} P_1(E')dE']$, (where the prime indicates the integration variable and E_0 is the injection energy into the acceleration process), equation (5) may be reduced to the normal form (Arkfen 1970):

$$\frac{d^2\tilde{N}(\eta, s)}{d\eta^2} + R(\eta, s)\tilde{N}(\eta, s) = \tilde{h}(\eta, s) , \qquad (6)$$

where

$$R(\eta, s) = g^2 P_2 \qquad (7)$$

and

$$\tilde{h}(\eta, s) = -g^2[\tilde{Q}(\eta, s) + N(\eta, 0)]/D . \qquad (8)$$

To solve equation (6) we follow the conventional route; first it is solved for the homogeneous part, and subsequently a particular solution is obtained. The solutions of the homogeneous part of equation (6) are obtained by the WKBJ method, provided the following criterion is fulfilled, (e.g., Mathews & Walker 1973):

$$|(dR/d\eta)/R^{3/2}| \ll 2 .$$

In Pérez-Peraza & Gallegos-Cruz (1994a) we discuss the conditions for which this inequality is satisfied in solar particle sources. The evaluation of this inequality gives us one of the options to determine the accuracy of the WKBJ method; two other alternatives are based, respectively, on the intrinsic relative error in the WKBJ solution (Bender & Orszag 1978) and on the error control function (Olver 1974) that we have considered in Appendixes A and C respectively. Now, according to Mathews & Walker (1973) if $P_2 > 0$ the solution of equation (6) is oscillatory, so that when the previous inequality is fulfilled the two independent solutions of the homogeneous part of equation (6) [when $P < 0$, and hence $R(\eta, s) < 0$] are given by $\tilde{N}_h = C_1 \tilde{N}_1(\eta, s) + C_2 \tilde{N}_2(\eta, s)$, where C_1 and C_2 are constants to be determined from the boundary conditions of the problem, and

$$\tilde{N}_{1,2} = R^{-1/4}(\eta, s) \exp\left\{\mp \int_{\eta_0}^{\eta} [R(\eta' s)]^{1/2} d\eta'\right\} = \exp\left[\mp \int_{E_0}^{E} P_2^{1/2}(E', s)dE'\right] g^{-1/2}(E, s) P_2^{-1/4}(E, s) . \qquad (9)$$

The particular solution of equation (6) may be obtained by the method of the Green function (e.g., Arfken 1970),

$$\tilde{N}_p(E, s) = \int_{E_0}^{E} \tilde{h}(E', s) G(E', E, s) dE' , \qquad (10)$$

where $E_0 \le E' \le E$, $G(E', E, s) = -\tilde{N}_1(E, s)\tilde{N}_2(E', s)/W(E', s)$ is the appropriate Green function of equation (6), and it is built with the homogeneous solutions given in equation (9). $W(E', s)$ is the Wronskian of $\tilde{N}_1$ and $\tilde{N}_2$.

Now, remembering that $R = g^2 P_2$ and $d\eta = g^{-1}dE$, the appropriated Green function for the variable E may be written as

$$G(E, E', s) = (-1/2)[g^{1/2}(E)P_2^{1/4}(E, s) g^{-1/2}(E') P_2^{1/4}(E', s)]^{-1} \exp\left[-\int_{E'}^{E} P_2^{1/2}(E'')dE''\right] . \qquad (11)$$

Formally, the general solution of equation (6) for the variable E is $\tilde{N}_G = C_1 \tilde{N}_1(E, s) + C_2 \tilde{N}_2(E, s) + \tilde{N}_p(E, s)$, where C_1 and C_2 can be evaluated from the boundary conditions of the problem. In the context of the evolution of cosmic particles the following conditions are required:

1. The spectrum decreases ($\tilde{N}_G \to 0$) as the energy increases ($\beta \to 1$).
2. $\tilde{N}_G$ tends to a constant value as the energy tends to the injection threshold value ($E \to E_0$).

Since $\tilde{N}_2$ is an increasing function of energy, it follows that $C_2 = 0$. Furthermore, it can be seen from equation (11) that both conditions (1) and (2) are completely satisfied by the Green function, so that within the frame of the energy spectrum of cosmic particles the two homogeneous solutions $\tilde{N}_{1,2}$ are not necessary in the general solution. Consequently $C_1 = C_2 = 0$, and the solution of equation (3) is entirely described by the particular solution. Substituting equations (8) and (11) into equation (10), the

WKBJ approximation for the general solution of equation (3) becomes

$$\tilde{N}_G(E, s) \simeq \tilde{N}_p(E, s) = \left(\frac{1}{2}\right) P_2^{-1/4}(E) \int_{E_0}^{E} \frac{[\tilde{Q}(E, s) + N(E', 0)]}{P_2^{1/4}(E')D(E')} \exp\left[-\frac{1}{2}\int_{E'}^{E} P_1\, dE'' - \int_{E'}^{E} P_2^{1/2} dE''\right] dE' \quad (12)$$

By introducing the explicit form of $P_2(E)$ we obtain

$$\tilde{N}_G(E, s) \simeq \frac{D^{1/4}(E)}{2[s + a(E)]^{1/4}} \int_{E_0}^{E} \frac{\{[q(E')/s] + N(E', 0)\}}{D^{3/4}(E')[s + a(E')]^{1/4}} \exp\left\{-\frac{1}{2}\int_{E'}^{E} P_1\, dE'' - \int_{E'}^{E} D^{-1/2}(E)[s + a(E'')]^{1/2} dE''\right\} dE' , \quad (12')$$

where $E_0 \leq E'' \leq E' \leq E$ and, using the assumption $Q(E, t) = q(E)$, so that $\bar{Q}(E, s) = q(E)/s$. To meet the requirements of the WKBJ method $a(E)$ must be a slowly varying function (which is the case for acceleration by MHD and Langmuir turbulence, as we will show later); hence, we can approximate $a(E'') \approx a(E') \approx a(E)$, or take their average values between E_0 and E (though this approximation is not necessary in the stationary case). Therefore, describing such an average value by a, equation (12′) becomes

$$N_G(E, t) \simeq \left(\frac{1}{2}\right) D^{1/4}(E) \int_{E_0}^{E} \frac{N(E', 0) \exp[-(1/2)\int_{E'}^{E} P_1\, dE'']}{D^{3/4}(E')} \mathscr{L}^{-1}\left\{\exp\left[-(s + a)^{1/2}\int_{E'}^{E} \frac{dE''}{D^{1/2}(E'')}\right] \Big/ (s + a)^{1/2}\right\} dE'$$
$$+ \left(\frac{1}{2}\right) D^{1/4}(E) \int_{E_0}^{E} \frac{q(E') \exp[-(1/2)\int_{E'}^{E} P_1\, dE'']\mathscr{L}^{-1}}{D^{3/4}(E')} \left\{\exp\left[-(S + a)^{1/2}\int_{E'}^{E} \frac{dE''}{D^{1/2}(E'')}\right] \Big/ s(s + a)^{1/2}\right\} dE' , \quad (13)$$

where $\mathscr{L}^{-1}$ indicates the Laplace inverse transform. Effectuating the Laplace transforms (see Appendix B) equation (13) becomes

$$N_G(E, t) \simeq \frac{D^{1/4}(E)}{(4\pi t)^{1/2}} \int_{E_0}^{E} \frac{N(E', 0)}{D^{3/4}(E')} \exp\left\{-at - \frac{1}{2}\int_{E'}^{E} P_1\, dE'' - \frac{1}{4t}\left[\int_{E'}^{E} D^{-1/2}(E'') dE''\right]^2\right\} dE'$$
$$+ \frac{D^{1/4}(E)}{(4\pi)^{1/2}} \int_0^t \frac{dt'}{t'^{1/2}} \int_{E_0}^{E} \frac{q(E')}{D^{3/4}(E')} \exp\left\{-at' - \frac{1}{2}\int_{E'}^{E} P_1\, dE'' - \frac{1}{4t'}\left[\int_{E'}^{E} \bar{D}^{1/2}(E'') dE''\right]^2\right\} dE' . \quad (14)$$

The method for carrying out the integration appearing in equation (14) is described in Appendix B. Hence, after integration in time we obtain the time-dependent WKBJ approximation for the general solution of equation (3), namely,

$$N(E, t) \simeq \frac{D^{1/4}(E)}{(4\pi)^{1/2}} \int_{E_0}^{E} \frac{\exp(-R_1)}{D^{3/4}(E')} \left[\frac{N(E', 0)}{t^{1/2}} \exp\left(-at - \frac{R_2}{t}\right) + \left(\frac{1}{2}\right)\left(\frac{\pi}{a}\right)^{1/2} q(E')R_3(E', E)\right] dE' , \quad (15)$$

where

$$R_1 = \tfrac{1}{2}\int_{E'}^{E} P_1\, dE'' , \qquad R_2 = \tfrac{1}{2}\int_{E'}^{E} D^{-1/2}(E'') dE'' ,$$

$$R_3 = [\mathrm{erf}(Z_1) - 1] \exp(2R_2 a^{1/2}) + [\mathrm{erf}(Z_2) + 1] \exp(-2R_2 a^{1/2}) ,$$

$$Z_{1,2} = (at)^{1/2} \pm R_2 t^{-1/2} ,$$

$$a = a(E) = (1/\tau) + 0.5[R_4(E) + R_4(E_0)] ,$$

and where (according to the definitions in eq. [4]),

$$R_4(E) = (dA/dE) - (d^2D/dE^2) .$$

It should be noted that by making $t \to \infty$ in equation (15), the first term becomes null and $[\mathrm{erf}(Z_2) + 1] \to 2.0$ in the second term (see eq. [B13] of Appendix B) and we obtain the corresponding stationary solution (eq. [24], below). Equation (15) gives the time-dependent energy spectrum of particles escaping with a probability τ^{-1} from the source, where they have been accelerated at the rates $A(E)$ and $D(E)$ (by any stochastic process) after an instantaneous injection $N(E, 0)$ and a continuous injection $q(E)$.

According to the precepts of the Laplace transforms the first term of equation (15) must be associated with a pulse in time (injection at $t = 0$), whereas the second term is associated with a continuous injection at $t \geq 0$. Therefore, each one of the terms appearing in equation (15) that determine the instantaneous particle energy spectrum at any time has a specific interpretation according to the initial conditions of the acceleration process, that is, according to the specific peculiarities of the astrophysical scenario. Hence, those two terms are mutually exclusive.

2.1.2. *Steady State Solution for any D(p)*

For the stationary solution, the left-hand term of equation (3) is null, so that by developing and arranging terms it can be rewritten as

$$\frac{d^2N}{dE^2} + P_1 \frac{dN}{dE} + P_2 N = f , \quad (16)$$

where P_1 and P_2 (with $s = 0$) have the same meaning as in equation (4) and $f = -q(E)/D(E)$. Equation (16) may be rewritten in normal form (Arfken 1970) as

$$\frac{d^2N(\eta)}{d\eta^2} + R(\eta)N(\eta) = h(\eta)\,, \tag{17}$$

where $R(\eta) = g^2P_2$ and $h(\eta) = g^2f$. If the criterion given in the inequality below equation (8) is fulfilled, the solutions of the homogeneous part of equation (17) when $R(\eta) < 0_2(P = -a/D < 0)$ may be obtained by the WKBJ method, yielding

$$N_{1,2}(\eta) = R(\eta)^{-1/4} \exp\left[\mp \int_{\eta 0}^{\eta} R(\eta')^{1/2} d\eta'\right]. \tag{18}$$

The particular solution of equation (17) may be built by the Green function method (or by the method of variation of parameters). In the variable E this solution is

$$N_p(E) = \int_{E_0}^{E} G(E, E')h(E')dE'\,, \tag{19}$$

where $G(E, E')$ is similar to $G(E, E', s)$ given in equation (11), but with $s = 0$.

Now, following the same arguments as in the time-dependent case (below eq. [11]), the Green function satisfies all the boundary conditions because it remains finite as $E \to \infty$ and is a constant when $E \to E_0$. Therefore, the homogeneous solutions are irrelevant, so that the solution of the steady state transport equation is entirely described by the particular solution, equation (19), which may be rewritten as

$$N(E) \simeq \frac{D^{1/4}(E)}{2a^{1/4}(E)} \int_{E_0}^{E} \frac{q(E')}{a^{1/4}(E')D^{3/4}(E')} \exp\left[-\frac{1}{2}\int_{E'}^{E} P_1\, dE'' - \int_{E'}^{E} a^{1/2}(E'')D^{-1/2}(E'')dE''\right] dE'\,. \tag{20}$$

Since the range of variation of $a(E)$ is short, we could eventually take again $a = \overline{a(E)} = (0.5)[R_4(E_0) + R_4(E)]$ (R_4 is the same as in § 2.1.1); however, this is not necessary (as it is in the time-dependent solution) but can be done for the sake of simplicity. With this simplification the abbreviated expression of the general solution, equation (20), becomes

$$N(E) \simeq \frac{D^{1/4}(E)}{2a^{1/2}} \int_{E_0}^{E} \frac{q(E')}{D^{3/4}(E')} \exp\left(-R_1 - 2a^{1/2}R_2\right)dE'\,, \tag{21}$$

where R_1 and R_2 are the same as defined below equation (15), in § 2.1.1. It can be seen that this equation is also directly obtained from equation (15) by letting $t \to \infty$.

It should be noted that up to equation (20) we have not made any assumption regarding the escape time, τ, which appears within the factor a in the time-dependent solution, equation (15), and the stationary solution, equation (20). In order to make comparisons with the work of other authors, we will consider in § 4 the classical assumption of τ = constant (mean confinement time) and $\tau \propto \beta^{-\mu}$ within the context of equations (15) and (20).

2.2. *Solutions for $D(p) \sim p^n/\beta$*

2.2.1. *Time-dependent Solution for $D(p) \sim p^n/\beta$*

Let us consider a momentum diffusion coefficient of the form $D(p) = (\alpha/3)p^n/\beta$, where α (= constant) is the parameter of turbulent acceleration efficiency. In this case the *systematic* and *diffusive* acceleration rates are, respectively,

$$A(E) = \frac{1}{p^2}\frac{\partial}{\partial p}[vp^2D(p)] = \left(\frac{\alpha}{3}\right)c(n+2)p^{n-1} = \left(\frac{\alpha}{3}\right)(n+2)c^{2-n}\beta^{n-1}\mathcal{E}^{n-1} \tag{22}$$

and

$$D(E) = 2v^2D(p) = 2(\alpha/3)c^{2-n}\beta^{n+1}\mathcal{E}^n = 2(\alpha/3)c^{2-n}(\mathcal{E}^2 - m^2c^4)^{(n+1)/2}/\mathcal{E}\,, \tag{23}$$

where $\mathcal{E} = E + mc^2$. Assuming an escape time $\tau^{-1} \propto \beta^{n-1}\mathcal{E}^{n-2} = \delta\beta^{n-1}\mathcal{E}^{n-2}$, where δ is a constant, the parameters $a(E)$ and P_2 defined in equations (4) are in this case,

$$a(E) = \delta\beta^{n-1}\mathcal{E}^{n-2} + (\alpha/3)c^{2-n}\mathcal{E}^{n-2}[(n-1)(3n+4)\beta^{n-3} - 2(n+1)\beta^{n-1} + 4\beta^{n+1}] \tag{24}$$

$$P_2(E, s) = \frac{s + a(E)}{D(E)} = \frac{\delta}{2(\alpha/3)c^{2-n}\beta^2\mathcal{E}^2} + \frac{1}{2\mathcal{E}^2}[(n-1)(3n+4)\beta^{-4} - 2(n+1)\beta^{-2} + 4] + \frac{s}{(\alpha/3)c^{2-n}\beta^{n+1}\mathcal{E}^n}$$

$$= \frac{1}{2(\alpha/3)c^{2-n}\beta^{n+1}\mathcal{E}^n}[s + a_n(E)]\,, \tag{25}$$

with $a_n(E) = (\alpha/3)c^{2-n}\beta^{n-1}\mathcal{E}^{n-2}[(d/\beta)^2 - 2(n+1)]$, $d = [(n-1)(3n+4) + \delta/(\alpha/3)c^{2-n}]^{1/2}$. Taking the average value of $a_n(E)$ between E_0 and E, namely a_n, introducing equations (22)–(25) into equation (13), effectuating the inverse Laplace transforms involved in that equation, and solving the integral in time (see Appendix B), we obtain

$$N(E, t) \simeq \frac{D^{1/4}(E)}{(4\pi)^{1/2}} \int_{E_0}^{E} \frac{\exp(-R_1)}{D^{3/4}(E')} \left[\frac{N(E', 0)}{t^{1/2}} \exp\left(-a_n t - \frac{R_2}{t}\right) + 0.5\left(\frac{\pi}{a_n}\right)^{1/2} q(E') R_3(E', E) \right] dE' , \tag{26}$$

where

$$R_1 = (1/2)\int^E P_1 dE'', \qquad P_1 = -[(A/D) - (2/D)(dD/dE)] , \qquad R_2 = (1/2)J_n ,$$

$$J_n = (3/2\alpha)^{1/2} c^{-(2-n)/2} \int_{E'}^{E} \beta''^{-(n+1)/2} \mathscr{E}''^{-n/2} dE'' ,$$

$$R_3(E', E) = [\mathrm{erf}(Z_1^n) - 1] \exp(2R_2 a_n^{1/2}) + [\mathrm{erf}(Z_2^n) + 1] \exp(-2R_2 a_n^{1/2}) ,$$

$$Z_{1,2} = (a_n t)^{1/2} \pm t^{-1/2} R_2$$

and $\mathscr{E}$ is the total energy of particles. Using the reasons mentioned below equation (11), equation (26) is the general WKBJ approximation for the general solution of equation (3) for any power n of the momentum in the momentum diffusion coefficient. Fermi-type acceleration corresponds to $D(p) \sim p^2/\beta$.

Equation (26) has a similar form to equation (15), such that when $t \to \infty$, the corresponding steady state expression is obtained. It should be noted that when $n = 2$ in equation (26) we obtain the solution given in equation (15).

2.2.2. *Steady State Solution for $D(p) \sim p^n/\beta$*

By assuming the same momentum diffusion coefficient as in § 2.2.1, the rates $A(E)$ and $D(E)$ and the parameter $a(E)$ are the ones given in equations (22)–(24) whereas P_2 is now,

$$P_2(E) = \frac{a(E)}{D(E)} = \frac{\delta}{(\alpha/3)c^{2-n}\beta^2\mathscr{E}^2} + \frac{1}{\beta^{n+1}\mathscr{E}^2}[(n-1)\beta^{n-3} + (n+1)\beta^{n-1} - 2\beta^{n+1}]$$

$$= \frac{\delta}{(\alpha/3)c^{2-n}\beta^2\mathscr{E}^2} + \left[\frac{(n-1)}{\mathscr{E}^2\beta^4} + \frac{(n+1)}{\mathscr{E}^2\beta^2} - \frac{2}{\mathscr{E}^2}\right] \approx \frac{1}{\beta^2\mathscr{E}^2}\left[\omega^2 + \frac{(n-1)}{\beta^2}\right] ,$$

where

$$\omega = [\delta/(\alpha/3)c^{2-n}] + (n+1) ,$$

and

$$P_2^{1/2}(E) = \left[\frac{a(E)}{D(E)}\right]^{1/2} \approx \frac{\omega}{\beta\mathscr{E}}\left[1 + \frac{(n-1)}{\omega^2\beta^2}\right]^{1/2} \approx \frac{\omega}{\beta\mathscr{E}}\left[1 + \frac{(n-1)}{2\omega^2\beta^2}\right] ,$$

the solution in this case is

$$N(E) \simeq \frac{D^{1/4}(E)}{2a^{1/4}(E)} \int_{E_0}^{E} \frac{q(E')}{a^{1/4}(E')D^{3/4}(E')} \left(\frac{\mathscr{E}}{\mathscr{E}'}\right) \left(\frac{\mathscr{E}'^2 - m^2c^4}{\mathscr{E}^2 - m^2c^4}\right)^{n/2} \left(\frac{\mathscr{E} + \beta\mathscr{E}}{\mathscr{E}' + \beta'\mathscr{E}'}\right)^{-[\omega + (n - 1/2\omega)]} \exp\left[-\frac{n-1}{2\omega}(\beta^{-1} - \beta'^{-1})\right] dE' . \tag{27}$$

It should be noted that when $n = 2$ in equation (27) we should obtain the solution given in equation (20); however, this is not exact due to the approximations applied in passing from equation (12′) to (13).

3. ESTIMATED ACCELERATION EFFICIENCIES

To apply the solutions developed in § 2 to the study of particle energy spectra, the establishment of the specific convective and diffusive energy change rates, $A(E)$ and $D(E)$, is required. As stated before, both rates depend on the momentum diffusion coefficient $D(p)$ which characterizes the interaction dynamics between particles and turbulence. Extensive work has been done in the literature related to momentum diffusion coefficients for different kinds of turbulence (e.g., Tverskoi 1967; Kulsrud & Ferrari 1971; Tsytovich 1977; Melrose 1980, 1986; Schlickeiser 1989; Dung & Schlickeiser 1990; Steinacker & Miller 1992). Here we reduce our study to specific applications of MHD and Langmuir turbulence, neglecting energy losses during the acceleration process because our goal is to compare the derived spectra by the WKBJ approximation with existing confident time-dependent spectra, which in general do not consider energy losses.

3.1. *Acceleration by MHD Turbulence*

When energy from MHD turbulence is transferred to particles by wave-particle resonant interactions (Fermi-like process), we have

$$D(p) = \alpha p^2/\beta . \tag{28}$$

By equating equation (28) with the diffusion coefficient $D(p) = \alpha p^2/3\beta$ (Parker & Tidman 1958), which appears in the Boltzmann equation when the scattering centers are considered to have a much larger mass than particles, the acceleration efficiency (when the turbulence is of the kind of the magnetosonic modes) is (e.g., Melrose 1986)

$$\alpha = 3(\pi/B)^2\langle k\rangle(V_\phi^2/c)\mathsf{W}\ln(\beta c/V_\phi)\,, \tag{29}$$

where V_ϕ is the phase velocity of the magnetosonic waves, β is the particle velocity in terms of the light speed,

$$\mathsf{W} = \int_{k_{\max}}^{k_{\min}} W_{\rm ms}(k', t)dk'\,,$$

is the instantaneous total energetic content of the turbulence (energy density per time unit), B is the average local magnetic field, and $\langle k\rangle$ is a characteristic wavenumber of the wavenumber spectrum, that can be calculated from (Achterberg 1981):

$$\langle k\rangle = W(t)^{-1}\int_{k_{\max}}^{K_{\min}} k' W_{\rm ms}(k', t)dk'\,,$$

with $W(t)$ = instantaneous turbulence level, so that for a Kolmogorov spectrum $[W_{\rm ms}(k, t) = W(t)k^{-5/3}]$, and then

$$\langle k\rangle = 2(k_m^2 k_M)^{1/3}\{[1 - (k_m/k_M)^{1/3}]/[1 - (k_m/k_M)^{2/3}]\}^{-1}\ {\rm cm}^{-1}\,, \tag{30}$$

where k_m and k_M are the minimum and maximum values of k.

A representative evaluation of α at the level of the sources is not an obvious task since W and $\langle k\rangle$ are determined by unprecise phenomenological criteria.

The estimation of W is hence usually based on the fact that in most of astrophysical plasma sources $\mathsf{W} \ll nKT$ (Kaplan & Tystovich 1973) and on some observational inferences (e.g., Smith & Breach 1989): $\langle k\rangle$ is estimated from the extreme values of the magnetosonic turbulence, k_m and k_M, which in turn are inferred from the extreme particle energy values according to the natural cutoff of the magnetosonic turbulence (e.g., Tverskoi 1967), and from the assumption that wave-particle interactions occur mainly in the range of wavelengths much larger than the particle gyroradius ($\lambda \gg \rho_g$).

The rates of wave particle energy interchange in the case of a Fermi-like process involving MHD turbulence are derived from the previous definitions of the systematic and dispersive rates as follows:

$$A(E) = \left\langle\frac{dE}{dt}\right\rangle = p^{-2}\frac{\partial}{\partial p}[p^2 vD(p)] = \left(\frac{4}{3}\right)\alpha pc = \left(\frac{4}{3}\right)\alpha\beta\mathscr{E} = \left(\frac{4}{3}\right)\alpha(E^2 + 2mc^2E)^{1/2} \tag{31}$$

$$D(E) = \langle dE^2/dt\rangle = 2v^2D(p) = (2/3)\alpha\beta^3\mathscr{E}^2 = (2/3)(E^2 + 2mc^2E)^{3/2}/(E + mc^2)\,. \tag{32}$$

3.1.1. *Acceleration Efficiency for Protons by Fast Magnetosonic Waves*

In the estimation of the efficiency α for acceleration of protons in a coronal flare source ($T = 10^7$, $n = 5 \times 10^9$ cm^{-3}, $B = 100$ G), it is considered, (Miller 1991) that wave-particle interactions occur with protons whose velocities are in the range $m_{p-A}v^2/2$ up to 10 GeV (where v_A is the Alfvén velocity) assuming $\lambda = 10\rho_g$, where $\rho_g = \beta m_p c^2/eB$, the values $k_m = 2\pi/\lambda_M$ and $k_M = 2\pi/\lambda_m$ are derived for 10 GeV protons and Alfvénic protons, respectively. Hence from equation (30) we obtain $\langle k\rangle = 3.2 \times 10^{-5}$ cm^{-1}. Taking $V_\phi \approx v_A$ for the fast magnetosonic mode, and assuming that W is continuously regenerated (i.e., it remains constant) with a value $\mathsf{W} = 0.1$ ergs cm^{-3} (e.g., Smith et al. 1989; Miller 1991), we obtain from equation (29) $0.045\ {\rm s}^{-1} \le \alpha \le 0.137\ {\rm s}^{-1}$ in the particle energy range (1–1000) MeV. In more realistic situations α must be probably lower than this estimated value due to the decrease of W by the effects of collisional wave damping. In fact, Miller et al. (1991) use $\alpha = 0.03\ {\rm s}^{-1}$ for evaluations of the proton energy spectrum.

3.1.2. *Acceleration Efficiency for Protons by Slow Magnetosonic Waves*

The eventual resonant acceleration of protons in coronal flares by the slow magnetosonic mode is disregarded because the strong damping of this mode by collisional processes. However, toward chromospheric levels the acceleration by the slow mode may become feasible as shown by Gallegos-Cruz et al. (1993). Nevertheless, we will not deal in this work with this option.

3.1.3. *Acceleration Efficiency for Electrons by Fast Magnetosonic Waves*

Acceleration of electrons is feasible by both magnetosonic modes, the fast and the low, in the regime of $\lambda \gg \rho_g$. For the interaction of electrons with the fast mode it should be remembered that the high-frequency cutoff for this type of turbulence occurs in the gyrofrequency of thermal protons, i.e., at $\omega_M \approx \Omega_p = 9.65 \times 10^3\, B$ Hz, (e.g., Braginskii 1965). Since $k_M = \omega_M/V_\phi$, hence in coronal sources $k_M = 3.13 \times 10^{-3}$ cm^{-1} when V_ϕ is evaluated in V_A, and so $\omega_M = 9.53 \times 10^5$ Hz; for electrons with a minimum energy, corresponding to the Alfvén velocity $\rho_g = 3.24 \times 10^2$ cm, so that for $\lambda_M = 10\rho_g$, $k_m = 2\pi/\lambda_M = 3.58$ cm^{-1} [corresponding to a frequency $\omega_m = 1.1 \times 10^9$ Hz, which is much higher than the upper cutoff frequency $\omega_M(\approx\Omega_p)$ for this kind of turbulence]. Assuming that interactions occur for wavelengths $\lambda \ge 10\rho_g$ (Miller 1991), i.e., $k_M \le 2\pi/10\rho_g$ ($\rho_g \le 2\pi/10k_M \approx 200$ cm) and using $\rho_g = \beta\mathscr{E}/eB$ it is found that wave particle interactions occur with electrons whose energies are $\beta\mathscr{E} \ge 6$ MeV, that is, electrons with $E \le 6$ MeV in coronal flare sources are excluded from the resonant interactions with the fast mode. To estimate α in this case let us assume an upper cutoff energy for electrons $E_{\max} = 500$ MeV, so that taking again $\lambda_M = 10\,\rho_g$, then $k_m = 2\pi/\lambda_M = 3.76 \times 10^{-5}$ cm^{-1}; consequently from equation (30) $\langle k\rangle = 2.67 \times 10^4$ cm^{-1}, and hence for the range (6–500) MeV we obtain from equation (29) $\alpha = 1.145\ {\rm s}^{-1}$ (with an error of 0.1%). This means that the fast magnetosonic turbulence is highly efficient in accelerating electrons but requires electrons with energies $\ge$6 MeV, which is highly restrictive at the level of solar sources even if they proceed from a preliminary acceleration stage.

3.1.4. *Acceleration Efficiency for Electrons by Slow Magnetosonic Waves*

Another alternative is the interaction of electrons with the slow magnetosonic mode (Gallegos-Cruz et al. 1993). In flare conditions the high-frequency cutoff is the same for both magnetosonic modes since it is determined by the gyrofrequency of thermal protons; however, the phase velocity of the slow mode is close to the velocity of sound ($V_\phi \approx V_s$), such that $k_M = \omega_M/V_s = 3.05 \times 10^{-2}$ cm^{-1}, and, hence, the interaction occurs for electrons with $\rho_g \leq 20.6$ cm, (i.e., for electrons with $\beta\mathscr{E} \geq 0.617$ MeV) corresponding to $E \geq 290$ keV, which is still a restrictive value in solar source conditions but easily satisfied when there is a preliminary acceleration stage. To evaluate $\langle k \rangle$ it can be assumed that $E_{\max} = 500$ MeV, which entails $\rho_g = 1.67 \times 10^4$ cm, so that if $\lambda_M = 10\rho_g$ then $k_m = 2\pi/10\rho_g = 3.76 \times 10^{-5}$ cm^{-1}, in such a form that $\langle k \rangle = 6.33 \times 10^{-4}$ cm^{-1}. Consequently, for electrons in the range (0.3–500 MeV) the efficiency remains practically constant (within a relative error <2%) with the value $\alpha = 0.042$ s^{-1}. This means that electrons of 300 keV need only a relatively modest acceleration efficiency value to be reaccelerated up to 500 MeV by interactions with the slow magnetosonic mode. It should be mentioned that at lower temperature and higher magnetic field strength (toward the chromosphere) the threshold energy for acceleration by the slow mode decreases drastically: for instance, for $T = 10^5$ and $B = 200$ G, even thermal electrons are able to be accelerated (Gallegos-Cruz et al. 1993).

3.2. *Acceleration by Langmuir Waves*

For resonant acceleration by Langmuir turbulence, the momentum diffusion coefficient may be expressed (e.g., Tystovich 1977; Melrose 1980) as

$$D(p) = \frac{2\pi e^2 \omega_p^2}{v^3} \int_{\omega_p/v}^{k_M} \frac{W(k)}{k^3}\, dk , \tag{33}$$

where ω_p is the plasma frequency and $W(k)$ is the energy spectrum of the turbulence (energy density per wavenumber k and time unit). Therefore, the corresponding rate is

$$A(E) = \left\langle \frac{dE}{dt} \right\rangle = \frac{1}{p^2} \frac{\partial}{\partial p} [p^2 v D(p)] = \frac{2\pi^2 e^2}{p} \left[\frac{m^2 c^4}{\mathscr{E}^2} W\left(\frac{\omega_p}{v}\right) + \frac{2\omega_p^2}{c^2} \int_{\omega_p/v}^{k_M} W(k) k^{-3} dk \right] ,$$

where $p = \beta\mathscr{E}/c$ is the particle momentum. Assuming $W(k) = W_0 k^{-5/2}$ (e.g., Borowsky & Eilek 1986), and taking $k_M \to \infty$ (e.g., Melrose 1980) we obtain

$$A(E) = \frac{\mathrm{K}_1 W_0}{\mathscr{E}} \left[\frac{m^2 c^4}{\mathscr{E}^2} \beta^{3/2} + \left(\frac{4}{9}\right) \beta^{7/2} \right] = K_L \left(\frac{\beta^{3/2}}{\mathscr{E}}\right) \left[1 - \left(\frac{5}{9}\right) \beta^2 \right] \text{ energy s}^{-1} \tag{34}$$

where

$$K_L = \mathrm{K}_1 W_0 = \frac{2\pi^2 e^2 c^{7/2}}{\omega_p^{5/2}} W_0 \qquad \text{energy}^2 \text{ s}^{-1} .$$

In a similar form, for the dispersive rate we have

$$D(E) = \left\langle \frac{dE^2}{dt} \right\rangle = 2v^2 D(p) = \frac{4\pi^2 e^2 \omega_p^2}{v} \int_{\omega_p/v}^{k_M} \frac{W(k)}{k^3}\, dk$$

considering again $W(k) = W_0 k^{-5/2}$ and $k_M \to \infty$, we obtain

$$D(E) = \frac{8\pi^2 e^2 c^{7/2}}{9\omega_p^{5/2}} W_0 \beta^{7/2} = \left(\frac{4}{9}\right) \mathrm{K}_1 W_0 \beta^{7/2} = \left(\frac{4}{9}\right) K_L \beta^{7/2} \qquad \text{energy}^2 \text{ s}^{-1} . \tag{35}$$

The energy density content of the turbulence, W_0, may be estimated from the relation $(U_{\rm tur}/U_{K_B T}) = 10^{-3}(n_e/T^3)^{1/2}$ (Rose et al. 1987), where $U_{K_B T} = nK_B T$ and $U_{\rm tur} = \int_{\omega_p/v}^{\infty} W_0 k^{-5/2} dk = (2/3) W_0 (v_{\rm th}/\omega_p)^{3/2}$. Using $v_{\rm th} = (2K_B T/m)^{1/2}$ and $\omega_p = (4\pi n e^2/m)^{1/2}$ we obtain $U_{\rm tur} = (2/3) W_0 (K_B T/2\pi n e^2)^{3/4}$ (energy cm^{-3}), so that $W_0 = 4 \times 10^{-9} n_e^{9/4}/T^{5/4}$ (eV cm^{-3}); therefore,

$$K_L = 7.45 \times 10^{-2} n_e/T^{5/4} \qquad \text{MeV}^2 \text{ s}^{-1} . \tag{36}$$

Equations (34)–(35) are valid through all the energy range of cosmic particles. Melrose (1980) derived the corresponding rates in the nonrelativistic and ultrarelativistic ranges.

4. THE INJECTION SPECTRUM

In order to compare our analytical spectra derived in § 2 with results of other authors, we have disregarded energy losses in the energy change rates. It should be remembered that effective particle acceleration requires that the acceleration rate be higher than the deceleration rate by collisional processes; hence, we are assuming a thin geometry in the source, so that energy losses during acceleration may be neglected. In this form there is only one restriction for effective acceleration by the linear wave-particle resonant process that requires that particles have $v \geq V_\phi$. Although this may be the case for electrons in very hot thermal plasmas, in general it is not the case for ions. A selective injection process is needed to feed the linear resonant process with some amount of particles, in order to be reaccelerated up to high energies. The problem of the injection is then narrowly related to the scenario of the global energetic particle generation phenomenon. A qualitative discussion about a possible scenario has been discussed by Pérez-Peraza & Gallegos-Cruz (1994b). For the injection process there are four different assumptions that fit within the frame of our analysis, two of them are of the monoenergetic kind, and the other two describing well-structured energy distributions:

(a) *Instantaneous Injection*:

$$N(E, 0) = N_0 \, \delta(E - E_0) \qquad \text{(particles of energy } E_0\text{)} \tag{37}$$

where N_0 denotes the number of impulsively injected particles in a pulse at $t = 0$.

(b) *Continuous Injection*:

$$q(E) = N_0 \, \delta(E - E_0) \qquad \text{(particles s}^{-1}\text{)} \tag{38}$$

(c) *Injection from a Thermal Population*:

$$q(E) = 2\pi N E^{1/2} \exp(-E/K_B T)/(\pi K_B T)^{3/2} \qquad \text{(particles per energy unit)} \tag{39}$$

where N is the number of particles per energy unit with $E \geq E_0$.

(d) *Injection by DC Fields from a Magnetic Neutral Current Sheet* (*MNCS*):
According to Pérez-Peraza et al. (1977, 1978) the energy spectrum from acceleration in a MNCS topology, as that given by Priest (1973), is

$$q(E) = 1.27 \times 10^4 N B_0 (m/n_e)^{1/2} (E_c^{1/4}/E^{3/4}) \exp[-1.12(E/E_c)^{3/4}] \qquad \text{particles eV}^{-1} \tag{40}$$

where N is the number of particles per energy unit with $E \geq E_0$, B_0 is the background magnetic field strength in the MNCS, $E_c = 1.34 \times 10^{11} [B_0^2 (m/n_e)^{1/2}]^{2/3}$ (eV), m is the particle mass, and n_e is the local number density. Hereafter, for evaluations of MNCS spectra we will assume that particles are injected from below to the acceleration region so that $n = 5 \times 10^{11}$ cm^{-3} and $B_0 = 500$ G.

5. APPLICATIONS: ANALYTICAL ENERGY SPECTRA FROM $D(p) \sim P^2/\beta$

To illustrate the applicability of the solutions derived in § 2, we have chosen acceleration rates that correspond to the two different kinds of turbulence analyzed in § 3, MHD and Langmuir waves. We have also considered the several assumptions discussed in § 4 about the injection process: monoenergetic injection, in which case we obtain analytical energy spectra. For thermal-type injection and injection from a MNCS, we obtain quasi-analytical spectra. According to the discussion at the end of § 2.1.1 the time-dependent spectrum describes both an instantaneous and a continuous injection, and the stationary spectrum describes only continuous injection.

The parameter $\mathcal{N}$ in $q(E)$ (eqs. [37]–[40]) appearing within the frame of $N(E', 0)$ in the first term of equation (15) indicates the total number of impulsively injected particles in a pulse a $t = 0$ (instantaneous injection) and will be denoted by N_0 (particles). When $\mathcal{N}$ appears within the frame of $q(E)$ in the second term of equation (15), then it indicates the number of particles injected per unit time (continuous injection at $t \geq 0$) and will be denoted by q_0 (particles s^{-1}).

Regarding the parameter τ, this is related to the fact that particles do not remain indefinitely in the region of the stochastic acceleration, but they disappear from the process either by escape or by nuclear transformation. The escape may be by spatial diffusion, convective, by drifts, etc. and may or may not be a particle energy dependent process. In the case of escape by spatial diffusion, it has been shown that it depends on the parallel spatial diffusion coefficient, $K_{/\!/}$ (Jokipii 1977; Dröge & Schlickeiser 1986; Dröge 1989): $\tau(E) = L^2/K_{/\!/} = 3L^2/lv$, where l is the mean free path between wave-particle collisions, and L is the size of the acceleration region. The last expression may be rewritten as $\tau(E) = K_2 \beta^{-1}$ in such a way that for typical parameters of a coronal flare ($T = 10^7$, $n = 5 \times 10^9$ cm^{-3}, and $L = 10^9$ cm) we obtain $K_2 = 3L^2/lv \simeq 1$, if $l \sim 10^8$ cm. To determine the effect that an energy-dependent confinement time, $\tau(E)$, has on the spectrum of the accelerated particles, we must consider this escape time within the solutions of the transport equation derived in § 2, particularly within the factor $a(E)$ defined below equations (15) and (20). To do so let the escape time be represented by $\tau(E) = K_2 \beta^{-\mu}$ such that $\mu = 0$ is equivalent to $\tau =$ constant and $\mu = 1$ describes spatial diffusion. In this work we analyze the case $\mu = 0$ and the case $\mu = 1$ (§§ 4.2.1–4.2.2). The cases $\mu \geq 2$ have been evaluated only by using numerical methods (e.g., Press et al. 1986).

5.1. *Spectra for MHD Turbulence, Monoenergetic Injection, and $\tau = cst.$, for $D(p) \sim p^2/\beta$*

5.1.1. *Time-dependent Spectrum for MHD Turbulence, Monoenergetic Injection, and $\tau = cst.$, for $D(p) \sim p^2/\beta$*

By using the MHD rates $A(E)$ and $D(E)$, $q(E) = q_0 \, \delta(E - E_0)$ in equation (15) and the properties of the delta function we obtain,

$$N(E, t) \simeq \frac{(\beta_0/\beta)^{1/4} (\mathscr{E}/\mathscr{E}_0)^{1/2} (\beta_0^{3/2} \mathscr{E}_0)^{-1}}{(4\pi\alpha/3)^{1/2}} \left[\left(\frac{N_0}{t^{1/2}} \right) \exp\left(-a_f t - \frac{3J_f^2}{4\alpha t} \right) + \left(\frac{q_0}{2} \right) \left(\frac{\pi}{a_f} \right)^{1/2} R_5(\mathscr{E}_0, \mathscr{E}) \right]$$

(particles per energy unit), (41)

where

$$R_5(\mathscr{E}_0, \mathscr{E}) = [\operatorname{erf}(Z_1) - 1] \exp[(3a_f/\alpha)^{1/2} J_f] + [\operatorname{erf}(Z_2) + 1] \exp[(-3a_f/\alpha)^{1/2} J_f] \, ;$$

$$Z_{1,2} = (a_f t)^{1/2} \pm (3a_f/4\alpha t)^{1/2} J_f \, ; \qquad a_f = (\alpha/3)(\bar{F} + 3/\alpha\tau) \, ,$$

$$\bar{F} = 0.5[\beta^{-1} + 3\beta - 2\beta^3 + \beta_0^{-1} + 3\beta_0 - 2\beta_0^3] \, ; \qquad \beta = (\mathscr{E}^2 - m^2c^4)^{1/2}/\mathscr{E} \, ;$$

and

$$J_f = \tan^{-1}\beta^{1/2} - \tan^{-1}\beta_0^{1/2} + 0.5 \ln \frac{(1 + \beta^{1/2})(1 - \beta_0^{1/2})}{(1 - \beta^{1/2})(1 + \beta_0^{1/2})} \, .$$

5.1.2. *Steady State Spectrum for MHD Turbulence, Monoenergetic Injection, and τ = cst., for $D(p) \sim p^2/\beta$*

Introducing $A(E)$ and $D(E)$ for MHD turbulence and the monoenergetic injection spectrum in equation (20) we obtain

$$N(E) \simeq (q_0/2)(a_f \alpha/3)^{-1/2}(\beta_0^{3/2}\mathscr{E}_0)^{-1}(\beta_0/\beta)^{1/4}(\mathscr{E}/\mathscr{E}_0)^{1/2} \exp\left[-(3a_f/\alpha)^{1/2}J_f\right] \quad \text{(particles per energy unit)}, \qquad (42)$$

where a_f and J_f are the same as in § 5.1.1.

5.2. *Spectra for MHD Turbulence, Monoenergetic Injection, and $\tau \propto 1/\beta$, for $D(p) \sim p^2/\beta$*

5.2.1. *Time-dependent Spectrum for MHD Turbulence, Monoenergetic Injection, and $\tau \propto 1/\beta$, for $D(p) \sim p^2/\beta$*

Let us take $\tau^{-1}(E) = \delta\beta$ (where $\delta = 1/K_2$ is a constant), which is the so-called case of escape by diffusion (e.g., Dröge & Schlickeiser 1986), so that the parameter a defined in equations (4) with the rates $A(E)$ and $D(E)$ for MHD turbulence becomes

$$a = \overline{a(E)} = (\alpha/3)[\beta^{-1} + \beta_0^{-1} + 3\beta(1 + \delta/\alpha) - 3\beta_0 - 2(\beta^2 + \beta_0^3)] .$$

Now, by substitution into equation (15) we obtain a similar spectrum to that given in equation (41) but with $\tau(E) = K_2 \beta^{-1}$.

5.2.2. *Steady State Spectrum for MHD Turbulence, Monoenergetic Injection, and $\tau \propto 1/\beta$, for $D(p) \sim p^2/\beta$*

In this case the parameter a of § 5.2.1 becomes $a(E) \approx (\delta + \alpha)\beta + (\alpha/3)\beta^{-1}$ so that

$$\frac{a(E)}{D(E)} = \frac{(3/\alpha)(\delta + \alpha)}{\beta^2\mathscr{E}^2} + (\beta^4\mathscr{E}^2)^{-1} = \frac{b^2}{\beta^2\mathscr{E}^2}\left[1 + \frac{1}{b^2\beta^2}\right],$$

with $b = [(3/\alpha)(\delta + \alpha)]^{1/2}$ and then $[a(E)/D(E)]^{1/2} = (b/\beta\mathscr{E}) + (1/2b\mathscr{E}\beta^3)$. Now introducing in equation (20)

$$(-1/2)\int_{E'}^{E} P_1\, dE'' = \ln(\beta/\beta') ,$$

and considering that

$$\exp\left[-\int_{E'}^{E} P_2\, dE''\right] = \exp\left\{-\int_{E'}^{E}\left[\frac{a(E')}{D(E')}\right]^{1/2} dE''\right\} = \left(\frac{\mathscr{E} + \beta\mathscr{E}}{\mathscr{E}' + \beta'\mathscr{E}'}\right)^{-(b+1/2b)} \exp\left[\left(\frac{-1}{2b}\right)(\beta^{-1} - \beta'^{-1})\right]$$

so that for a monoenergetic injection as that given in equation (38), then the energy spectrum, equation (20) or (21), becomes

$$N(E) = \frac{(q_0/2)(\beta_0/\beta)^{1/4}(\mathscr{E}/\mathscr{E}_0)^{1/2}}{(\alpha/3)^{1/2}a^{1/4}(E)a^{1/4}(E_0)\beta_0^{3/2}\mathscr{E}_0}\left[\frac{\mathscr{E} + \beta\mathscr{E}}{\mathscr{E}_0 + \beta_0\mathscr{E}_0}\right]^{-(b+1/2b)} \exp\left[\left(\frac{-1}{2b}\right)(\beta^{-1} - \beta_0^{-1})\right] \quad \text{(particles per energy unit)}. \qquad (43)$$

5.3. *Spectra for Langmuir Turbulence, Monoenergetic Injection, and τ = cst*

5.3.1. *Time-dependent Spectrum for Langmuir Turbulence, Monoenergetic Injection, and τ = cst*

In this case $D(p)$ is that given in equation (33), so that using the rates $A(E)$ and $D(E)$ given in equations (33)–(35) into equation (15) we obtain the following analytical spectrum:

$$N(E, t) \simeq \frac{3(\mathscr{E}/\mathscr{E}_0)^{1/2}}{4(\pi K_L)^{1/2}\beta_0^{1/4}\beta^{3/2}}\left[t^{-1/2}N_0 \exp\left(-a_L t - \frac{9J_L^2}{16K_L t}\right) + \left(\frac{q_0}{2}\right)\left(\frac{\pi}{a_L}\right)^{1/2}\right.$$
$$\left.\times\left\{[\text{erf}(Z_1) - 1]\exp\left[\left(\frac{3}{2}\right)\left(\frac{a_L}{K_L}\right)^{1/2} J_L\right] + [\text{erf}(Z_2) + 1]\exp\left[\left(\frac{-3}{2}\right)\left(\frac{a_L}{K_L}\right)^{1/2}\right]\right\}\right] \quad \text{(particles per energy unit)}, \qquad (44)$$

where

$$J_L = F(\beta) - F(\beta_0)\,, \quad F(\beta) = mc^2\left[\frac{\beta^{1/4}}{(1 - \beta^2)^{1/2}} + \frac{3\sin^{-1}\beta}{4\beta^{3/4}} + \left(\frac{9}{4}\right)\beta^{1/2} + \left(\frac{9}{216}\right)\beta^{9/4}\right],$$
$$a_L = \overline{a_L(E)} = (1/\tau) + 4.5K_L[H(\beta) + H(\beta_0)]\,, \qquad H(\beta) = \mathscr{E}^{-1}\,|-3.22\beta^{7/2} + 4.17\beta^{3/2} - \beta^{-1/2}|\,,$$
$$Z_{1,2} = (a_L t)^{1/2} \pm 3J_L/4(K_L t)^{1/2}\,,$$

and K_L is the parameter of acceleration efficiency defined in § 3.

5.3.2. *Steady State for Langmuir Turbulence, Monoenergetic Injection, and τ = cst*

Introducing the acceleration rates for Langmuir turbulence, equations (34)–(35), that were derived from the diffusion coefficient given by equation (33), into equation (20), we obtain

$$N(E) \simeq \frac{3q_0(\mathscr{E}/\mathscr{E}_0)^{1/2}}{4(a_L K_L)^{1/2}\beta_0^{1/4}\beta^{3/2}} \exp\left[\frac{-3}{2}\left(\frac{a_L}{K_L}\right)^{1/2} J_L\right] \quad \text{(particles per energy unit)}. \qquad (45)$$

5.4. *Spectra for MHD Turbulence, Thermal-Type Injection, and τ = cst., for $D(p) \sim p^2/\beta$*

5.4.1. *Time-dependent Spectrum for MHD Turbulence, Thermal-Type Injection, and τ = cst., for $D(p) \sim p^2/\beta$*

Introducing the thermal injection spectrum equation (39), into equation (15), we obtain,

$$N(E, t) \simeq \frac{D^{1/4}(E)}{\pi(K_B T)^{3/2}\beta} \int_{E_0}^{E} \frac{E'^{1/2}\beta' e^{-E'/k_B T}}{D^{3/4}(E')} \left[\frac{N_0 e^{-(a_f t + 3J_f^2/4\alpha t)}}{t^{1/2}} + \frac{q_0}{2}\left(\frac{\pi}{a_f}\right)^{1/2} R_5(E', E)\right] dE' \quad \text{(particles per energy unit)}, \quad (46)$$

where K_B is the Boltzmann constant, R_5 was defined below equation (15), and a_f was defined below equation (41).

5.4.2. *Steady State Spectrum for MHD Turbulence, Thermal-Type Injection, and τ = cst., for $D(p) \sim p^2/\beta$*

Substitution of the thermal spectrum and the MHD acceleration rates into equation (21) gives

$$N(E) \simeq \frac{(q_0/2)D^{1/4}(E)}{(\pi a_f)^{1/2}(K_B T)^{3/2}\beta} \int_{E_0}^{E} \frac{E'^{1/2}\beta'}{D^{3/4}(E')} \exp\left[\frac{-E'}{K_B T} - \left(\frac{3a_f}{\alpha}\right)^{1/2} J_f(E, E')\right] dE' \quad \text{(particles per energy unit)}, \quad (47)$$

where a_f has been previously defined, and $J_f(E, E')$ is the same as $J_f(E, E_0)$ with $E' = E_0$.

5.5. *Spectra for Langmuir Turbulence, Thermal-Type Injection, and τ = cst*

5.5.1. *Time-dependent Spectrum for Langmuir Turbulence, Thermal-Type Injection, and τ = cst*

Introducing the acceleration rates for Langmuir turbulence in equation (15) we obtain

$$N(E, t) \simeq \frac{D^{1/4}(E)}{\pi(K_B T)^{3/2}} \int_{E_0}^{E} \frac{E'^{1/2}(\mathscr{E}/\mathscr{E}')^{1/2}(\beta'/\beta)^{19/8}}{D^{3/4}(E')} \exp\left(\frac{-E'}{K_B T}\right)\left[\frac{N_0}{t^{1/2}} \exp\left(\frac{-a_L t - 9J_L^2}{16K_L t}\right) + \left(\frac{q_0}{2}\right)\left(\frac{\pi}{a_L}\right)^{1/2}\left\{[\operatorname{erf}(Z_1) - 1]\right.\right.$$

$$\left.\left.\times \exp\left[\left(\frac{3}{2}\right)\left(\frac{a_L}{K_L}\right)^{1/2} J_L\right] + [\operatorname{erf}(Z_2) + 1] \exp\left[\left(\frac{-3}{2}\right)\left(\frac{a_L}{K_L}\right)^{1/2} J_L\right]\right\}\right] dE' \quad \text{(particles per energy unit)}, \quad (48)$$

where a_L, K_L, and J_L are similar that in § 5.3.1.

5.5.2. *Steady State Spectrum for Langmuir Turbulence, Thermal-Type Injection, and τ = cst*

By substitution of the Langmuir acceleration rates $A(E)$ and $D(E)$ into equation (21) we obtain

$$N(E) \simeq \frac{q_0 D^{1/4}(E)}{(\pi a_L)^{1/2}(K_B T)^{3/2}} \int_{E_0}^{E} \frac{E'^{1/2}(\mathscr{E}/\mathscr{E}')^{1/2}(\beta'/\beta)^{19/8}}{D^{3/4}(E')} \exp\left[\left(\frac{-E'}{K_B T}\right) - \left(\frac{3}{2}\right)\left(\frac{a_L}{K_L}\right)^{1/2} J_L\right] dE' \quad \text{(particles per energy unit)}. \quad (49)$$

5.6. *Spectra for MHD Turbulence, MNCS Injection, and τ = cst*

5.6.1. *Time-dependent Spectrum for MHD Turbulence, MNCS Injection, and τ = cst*

Introducing the spectrum from MNCS injection, equation (40), into equation (15) we obtain

$$N(E, t) \simeq \frac{1.07 \times 10^{-5} D^{3/4}(E)}{(4\pi)^{1/2}\beta} \int_{E_0}^{E} \frac{\beta' e^{-1.12(E'/E_c)^{3/4}}}{E'^{3/4} D^{3/4}(E')} \left[\frac{N_0 e^{-a_f t - 3J_f^2/4\alpha t}}{t^{1/2}} + \frac{q_0}{2}\left(\frac{\pi}{a_f}\right)^{1/2} R_5(E', E)\right] dE' \quad \text{(particles eV}^{-1}\text{)}. \quad (50)$$

5.6.2. *Steady State Spectrum for MHD Turbulence, MNCS Injection, and τ = cst*

Introducing the injection spectrum given in equation (40) into equation (21), we obtain

$$N(E) \simeq \frac{1.07 \times 10^{-5}(q/2)D^{3/4}(E)}{\beta a_f^{1/2}} \int_{E_0}^{E} \frac{\beta' \exp[-1.12(E'/E_c)^{3/4} - (3a_f/\alpha)^{1/2} J(E', E)]}{E'^{3/4} D^{3/4}(E')} dE' \quad \text{(particles eV}^{-1}\text{)}. \quad (51)$$

5.7. *Spectra for Langmuir Turbulence, MNCS Injection, and τ = cst*

5.7.1. *Time-dependent Spectrum for Langmuir Turbulence, MNCS Injection, and τ = cst*

By substitution of the rates given in equations (34)–(35), and the injection spectrum from a MNCS, equation (40), into equation (15) we obtain

$$N(E, t) \simeq \frac{1.07 \times 10^{-5} D^{1/4}(E)}{(4\pi)^{1/2}} \int_{E_0}^{E} \frac{(\mathscr{E}/\mathscr{E}')^{1/2}(\beta'/\beta)^{19/8}}{D^{3/4}(E')E'^{3/4}} \exp\left[-1.12\left(\frac{E'}{E_c}\right)^{3/4}\right]\left[N_0 t^{-1/2} \exp\left(\frac{-a_L t - 9J_L^2}{16K_L t}\right) + \left(\frac{q_0}{2}\right)\left(\frac{\pi}{a_L}\right)^{1/2}\right.$$

$$\left.\times \left\{[\operatorname{erf}(Z_1) - 1] \exp\left[\left(\frac{3}{2}\right)\left(\frac{a_L}{K_L}\right)^{1/2} J_L\right] + [\operatorname{erf}(Z_2) + 1] \exp\left[\left(\frac{-3}{2}\right)\left(\frac{a_L}{K_L}\right)^{1/2} J_L\right]\right\}\right] dE' \quad \text{(particles eV}^{-1}\text{)}. \quad (52)$$

5.7.2. *Steady State Spectrum for Langmuir Turbulence, MNCS Injection, and τ = cst*

Introducing the Langmuir acceleration rates and the MNCS injection spectrum given in equation (40), we obtain for this case

$$N(E) \simeq \frac{1.07 \times 10^{-5}(q_0/2)D^{1/4}(E)}{a_L^{1/2}} \int_{E_0}^{E} \frac{(\mathscr{E}/\mathscr{E}')^{1/2}(\beta'/\beta)^{19/8}}{D^{3/4}(E')E'^{3/4}} \exp\left[-1.12\left(\frac{E'}{E_c}\right)^{3/4} - \left(\frac{3}{2}\right)\left(\frac{a_L}{K_L}\right)^{1/2} J_L\right] dE' \quad (\text{particles eV}^{-1}) . \quad (53)$$

6. RESULTS

The results are illustrated in Figures (1)–(10). Figure (1) shows the comparison of the steady state WKBJ solution, given in equation (21) (*circles*) for monoenergetic injection with the nonrelativistic analytical solution given by Ramaty (1979) (*dashed line*), the numerical solution given by Miller et al. (1990) (*solid line*), and the results of a Monte Carlo simulation of Miller et al. (1987) (*crosses*). The normalization to the Miller et al. results was done at the point of minimum energy (1 MeV) with a normalization factor <0.05 (5%) as determined directly from the figure of the cited work.

It can be seen that the WKBJ equilibrium spectrum coincides quite correctly with the numerical spectrum. Figure 2 shows the comparison of the steady state WKBJ solution (*circles*) with the ultrarelativistic solution for $E_0 \gg mc^2$ obtained by Ramaty (1979) (*solid line*) and the ultrarelativistic solution for $\beta = 1$ obtained in equation (B13) of this work (*dashed line*). It can be observed that the asymptotic solution with $E_0 \gg mc^2$ fits quite correctly the overall solution by the WKBJ method for $E \gg 10^4$ MeV, becoming identical for injection energy values $E \geq 10^5$ MeV, whereas the solution (B13), for $\beta = 1$, requires much higher values of E_0 to approach the overall WKBJ solution.

Figure 3 shows time-dependent proton spectra for monoenergetic injection: the open circles represent the WKBJ solution with continuous injection (second term of eq. [41]), and the solid line is the corresponding numerical solution of Miller et al. (1990). The comparison of both energy spectra is done for three different acceleration times, with $\alpha = 0.04$ s^{-1} and $\tau = 1$ s. Figure 4 shows the same solutions for six different acceleration times and different values of the acceleration parameters ($\alpha = 0.2$ s^{-1}, $\tau = 0.2$ s). It should be noted that the discrepancy at high energies between the WKBJ and the numerical solution is more important with the parameters used in Figure 4. This sensitivity to the acceleration parameters at high energies may be attributed to the approximation done in passing from equation (12′) to equation (13), and of course, to the approximate nature of the method. In fact, in the case of the steady state solution where the mentioned approximation was not done, there is not such a notorious discrepancy at high energies, as can be observed in Figure 1.

Figure 5 shows the time-dependent energy spectra of protons which have been accelerated by MHD turbulence while injected continuously from a MNCS for five different acceleration times. These curves show how the spectra reach equilibrium as time elapses. Within an error of $\sim 1\%$ we have determined that the time for particles of $E \leq 10$ MeV to reach equilibrium is $t \geq 4$ s; for particles of $E \leq 10^2$ MeV it is $t \geq 7$ s; for $E \leq 10^3$ MeV it is $t \geq 13$ s; and for $E \leq 10^4$ MeV it is $t \geq 22$ s. Figure 6 shows similar results to those of Figure 5 but for different acceleration parameters α, τ. In this case the equilibrium for particles of $E \leq 10$ MeV is

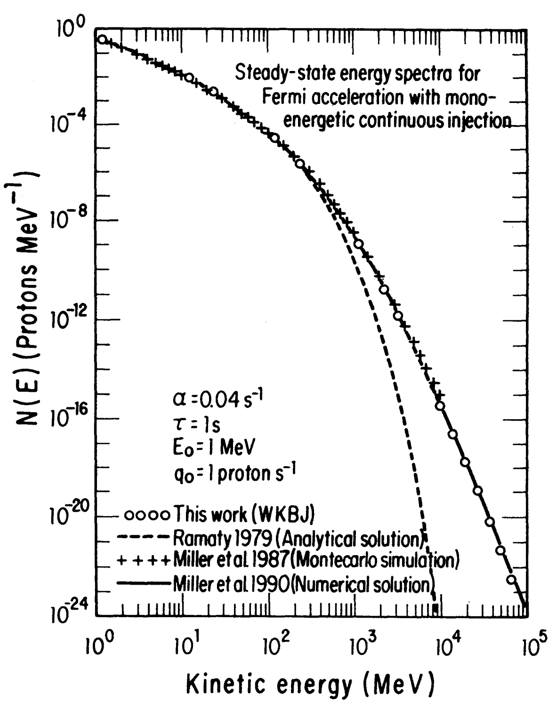

FIG. 1

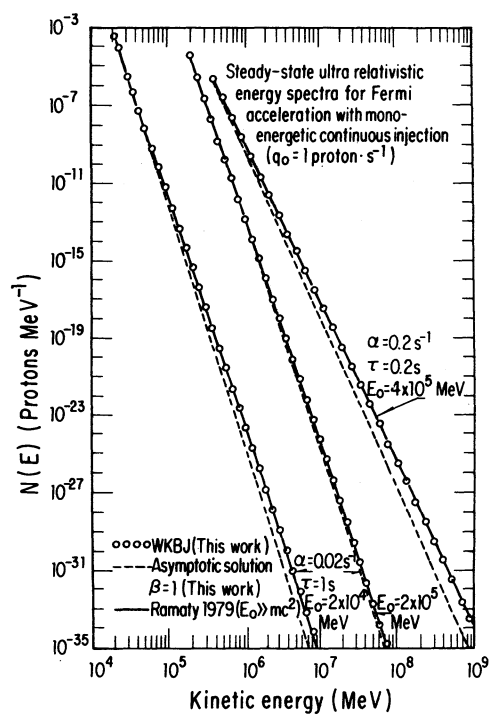

FIG. 2

FIG. 1.—Steady state spectra for Fermi acceleration with monoenergetic injection

FIG. 2.—Ultrarelativistic steady state spectra for Fermi acceleration with monoenergetic injection

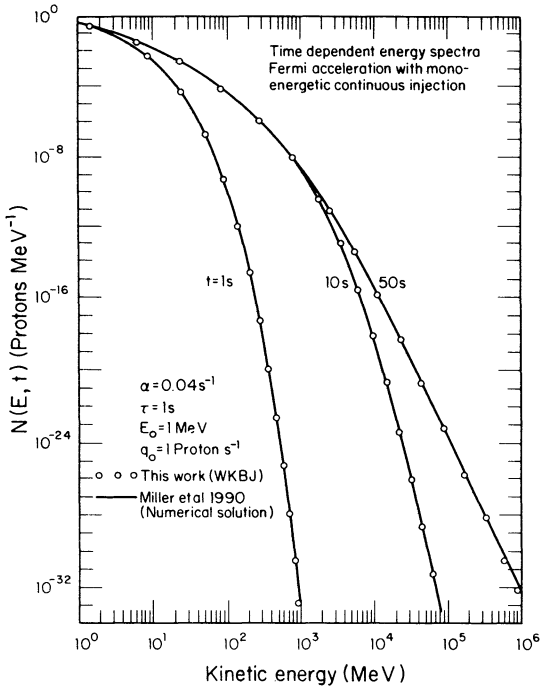

FIG. 3

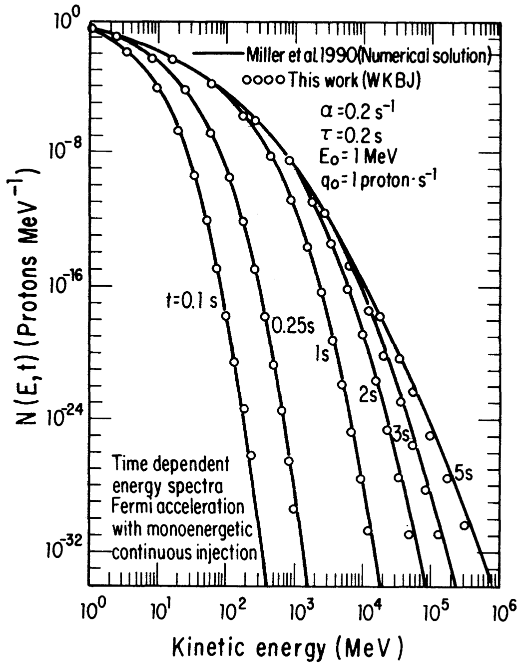

FIG. 4

FIG. 3.—Time-dependent energy spectra for Fermi acceleration with monoenergetic continuous injection, at three different acceleration times

FIG. 4.—Time-dependent energy spectra for Fermi acceleration with monoenergetic continuous injection, at six different acceleration times

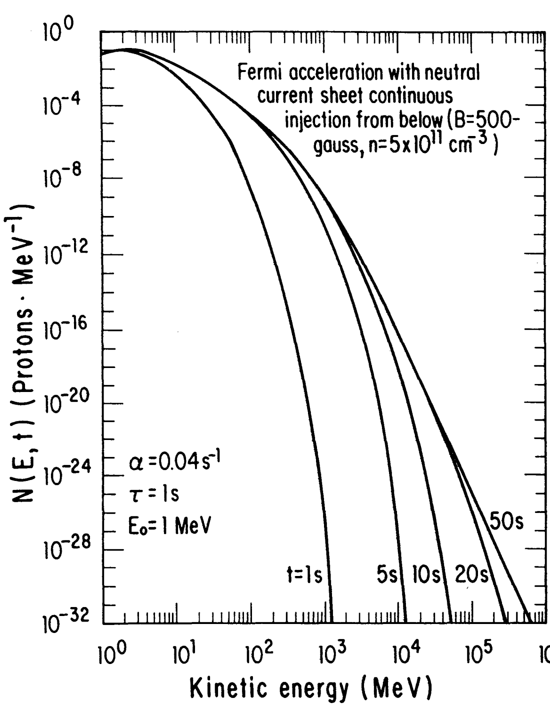

FIG. 5

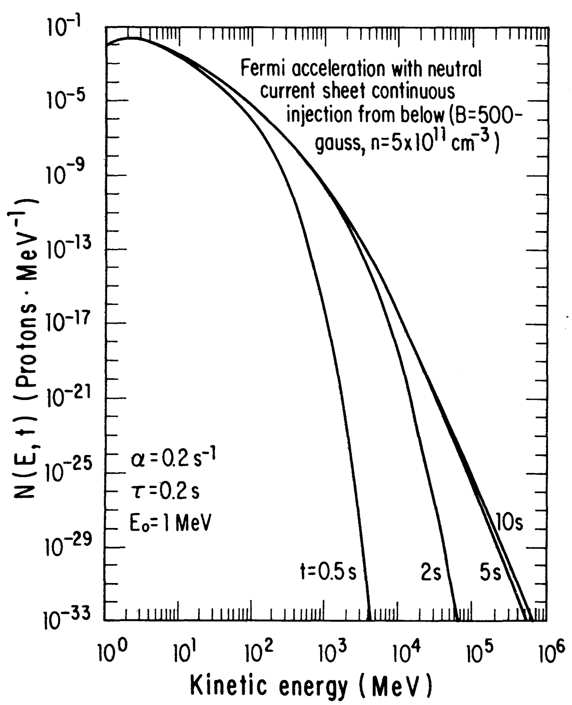

FIG. 6

FIG. 5.—Time-dependent energy spectra for Fermi acceleration with magnetic neutral current sheet continuous injection, at five different acceleration times

FIG. 6.—Time-dependent energy spectra for Fermi acceleration with magnetic neutral current sheet continuous injection, at four different acceleration times

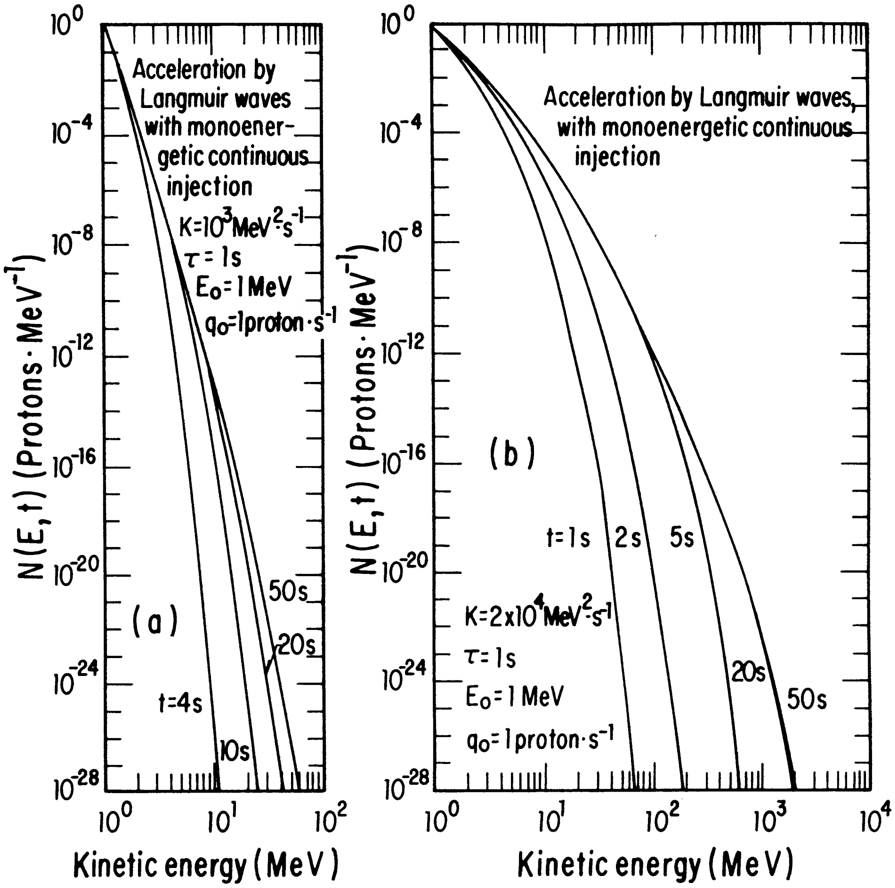

FIG. 7.—(*a*) Time-dependent energy spectra for acceleration by Langmuir turbulence with monoenergetic continuous injection, at four different acceleration times. (*b*) Same as (*a*), for different acceleration parameters and times.

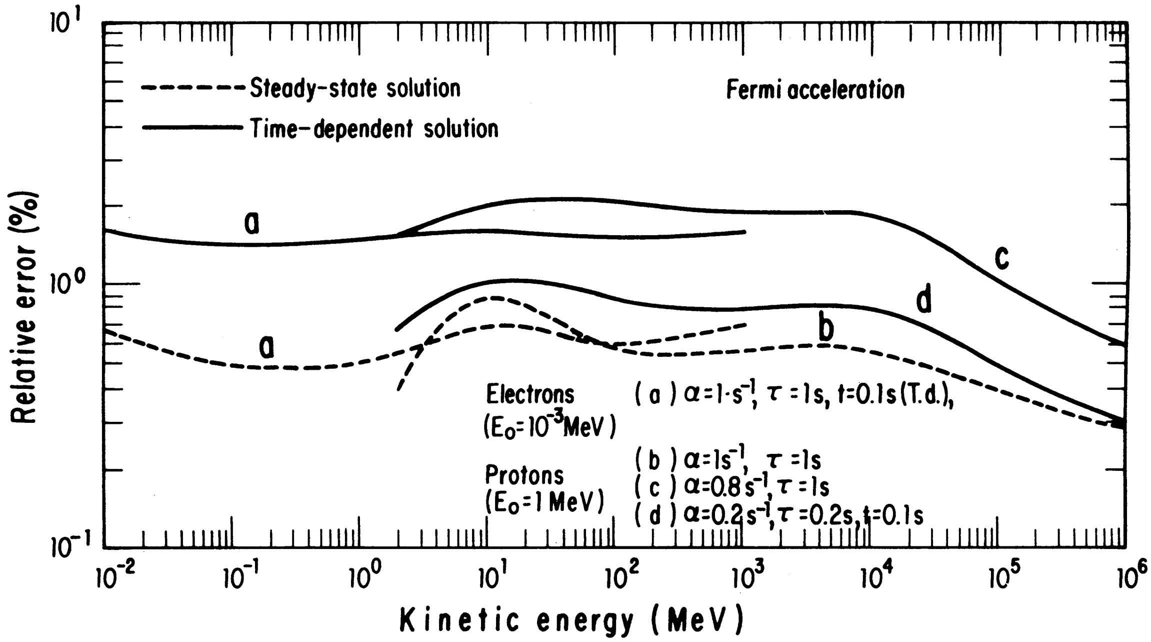

FIG. 8.—Intrinsic relative error of the WKBJ method, for the equilibrium and time-dependent solutions of eq. (3), and for protons and electrons

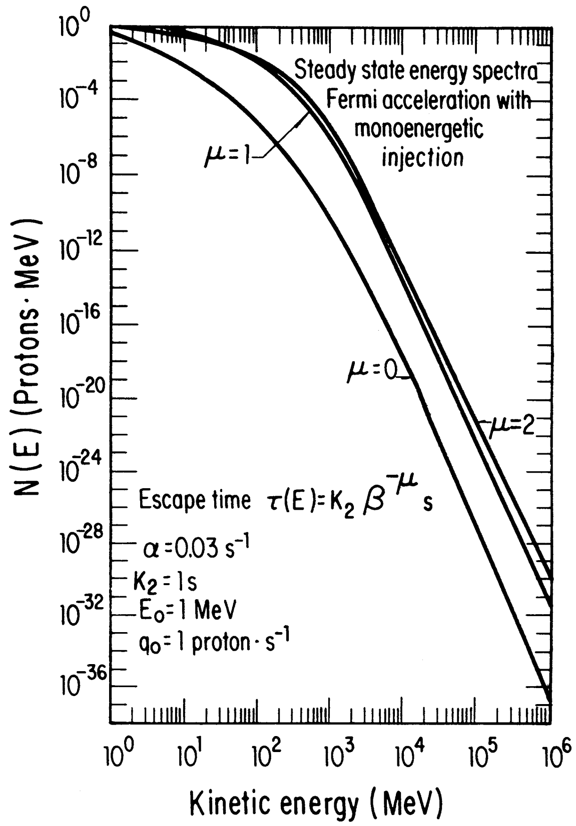

FIG. 9

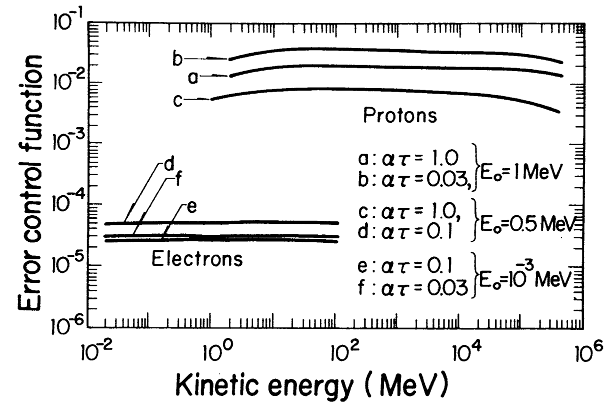

FIG. 10

FIG. 9.—Equilibrium energy spectra for Fermi acceleration with monoenergetic injection, for different velocity dependences of the escape time τ

FIG. 10.—Error control function of the WKBJ method for different values of the parameters $\alpha\tau$ in the steady state spectra of Fermi-type acceleration by MHD turbulence.

reached when $t \geq 0.9$ s; for $E \leq 10^2$ MeV when $t \geq 1.5$ s; for $E \leq 10^3$ MeV when $t \geq 2.6$ s; and for $t \leq 10^4$ when $t \geq 5$ s. Figures 7*a* and 7*b* show the time-dependent energy spectra of protons that have been accelerated by Langmuir turbulence while injected continuously with a monoenergetic flux, for four different acceleration times. A different value of the acceleration efficiency K_L is assumed in Figure 7*b* relative to Figure 7*a*. For the acceleration parameters used in Figure 7*a*, the equilibrium for particles of $E \leq 10$ MeV is reached when $t \geq 21$ s; for $E \leq 40$ MeV when $t \geq 38$ s; for $E \leq 10^3$ MeV when $t \geq 98$ s. In the case of the parameters employed in Figure 7*b* particles of $E \leq 10$ MeV reach the equilibrium when $t \geq 5.4$ s; for $E \leq 10^2$ MeV when $t \geq 11$ s; and for $E \leq 10^3$ MeV it is $t \geq 21$ s.

Figure 8 shows an estimation of the accumulated relative error S_2 (in absolute value), when the WKBJ series [eq. (A3)] is cut at the second equation of the series: the solution of these two first equations determines the solution of the homogeneous part of equation (6). A representative evaluation of the maximum relative error is obtained by taking the maximum values of α and τ within the function $(\alpha/\bar{a})^{1/2}$ in the stationary case (*dashed lines*) appearing in equation (57) and the function $(\alpha/\bar{a})^{1/2} \exp(-\bar{a}t)$ in the time-dependent case (*solid lines*) appearing in equation (56). This "maximization" may not include the parameter E_0 since any change in its value within the integral appearing in equations (56) and (57) is practically compensated by the opposite effect in the term $[1 - (mc^2/\mathscr{E}_0)^2]$. The curves (a) show the maximum relative error for electrons, and the curves (b), (c), and (d) for protons. In particular, the parameters used in curve (d) (α, τ, E_0) correspond to those used in Figure 4. It can be observed that in all cases, both steady state and time-dependent, the absolute value of the relative error is always lower than 3%.

Figure 9 shows the effect of the velocity dependence of the escape time, τ, on the energy spectrum when protons are accelerated by MHD turbulence while continuously injected into the stochastic acceleration process. It can be seen that the curves with $\mu \neq 0$ tend toward a power-law from transrelativistic energies ($E \geq mc^2$) up to the ultrarelativistic energy range. This may be interesting in the light of new observational results (Mandzhavidze et al. 1993). However, it should be mentioned that the difference between the spectra with $\mu = 0$ (τ = constant) and $\mu \neq 0$ (velocity-dependent) is rather qualitative.

Figure 10 shows the absolute value of the *error control function* (Olver 1974), that is the accumulated error up to the energy E due to the approximations made in the functions $R(\eta, s)$ or $R(\eta)$ (appearing in the normal form of the differential eq. [3]), that we have used to construct the approximated solution by the WKBJ method. This is illustrated for a wide range of the product $\alpha\tau$ (Appendix C). It can be appreciated from this figure that the error control function is very small, which is consistent with the relative error (Bender & Orszag 1978) of the WKBJ solution that we have evaluated in this work.

7. ANALYSIS AND CONCLUSIONS

Particle acceleration in a wave-particle interaction process and the energetic evolution of particles are two correlated problems whose study leads to the establishment of a noncollisional transport equation for the distribution function of particles. Within the context of particle and turbulence isotropy, the transport equation reduces to a diffusion equation in momentum space which in turn may be rewritten as a Fokker-Planck type equation in energy space. The latter one has been generalized as a continuity equation which includes, in addition to the dynamic effects of wave-particle interactions, the additional effects that energetic particles may undergo in a plasma. Until now the *time-dependent* solution of the transport equation *in the entire energetic range* may be solved only by numerical methods or Monte Carlo simulations; analytical solutions exist only in limited energy ranges. In this work we use the WKBJ method and the Laplace transform technique to obtain an approximate analytical solution throughout the entire particle energy domain ("WKBJ solution"), and we show that the existing closed solutions may be obtained as particular cases of the WKBJ solutions. The analysis of the obtained results may be carried out in three main directions: (1) Reliability of the WKBJ solutions, (2) equilibrium tendency of particle acceleration by plasma turbulence, and (3) effects of particle escape on the energetic particle distribution.

1. To evaluate the reliability of the time-dependent and stationary energy spectra obtained with the WKBJ method, three alternatives may be considered: namely (*a*) the comparison of $N(E, t)$ and $N(E)$ with existing results derived by other methods, (*b*) evaluation of the intrinsic relative error of the WKBJ method according to Appendix A (Bender & Orszag 1978), and (*c*) evaluation of the error control function (Olver 1974) according to Appendix C. These three options have been done for Fermi type acceleration by MHD turbulence:

(*a*) Four methods have been used for the comparison of the WKBJ solutions. These are (i) analytical closed solutions in the ranges $E_0 \ll mc^2$ and $E_0 \gg mc^2$, (ii) Monte Carlo simulations, (iii) numerical solutions, and (iv) asymptotic solutions for $\beta = 1$. For the sake of such comparisons we have assumed continuous monoenergetic injection, constant escape time and MHD turbulence. As can be appreciated in the figures, there is an excellent agreement, for both the steady state and the time-dependent spectra, between the global (for any β value) WKBJ solution and the analytical closed solutions, and the Monte Carlo solutions.

A slight discrepancy between the WKBJ solution and the numerical spectra may be noted in the high-energy extreme of the spectra, when $\alpha\tau \geq 0.04$. In the steady state case this discrepancy may be attributed to two main sources: the approximate nature of the WKBJ method and probable deficiencies of convergence in the numerical results that we used as reference frame. In fact, most of numerical methods present convergence problems as the particle energy increases.

For the time-dependent case, an additional source of discrepancy at high energies is the approximation made in $a(E)$ in passing from equation (12′) to equation (13). Regarding the comparison to the asymptotic solution, it can be seen in Figure 2 that good agreement is obtained only when the threshold energy $E_0 \gg 10^5$ MeV, whereas a perfect agreement between the WKBJ solution and the closed solution (for $E_0 \gg mc^2$) is obtained with $E_0 \geq 4 \times 10^4$ MeV.

(*b*) The relative error of the WKBJ method is estimated on basis of the first two terms of the series given by equation (A3) of Appendix A, in such a form that the evaluation of the third term gives the relative error, S_2, as indicated in equation (A5). Therefore, to quantify the precision of our results, let us consider equation (7): $R(\eta, s) = g^2 P_2 = R[\eta(E_0, E), s] = f(E, s)$; since $f(x)$ in equation (A2) corresponds to the coefficient of $N(\eta, s)$ in equation (6), or $N(\eta)$ in equation (17), we can set $f(x) = f(E, s)$. To evaluate $f(x)$ we use the rates $A(E) = 4\alpha\beta\mathscr{E}/3$ and $D(E) = \alpha\beta^3\mathscr{E}^2/3$ in P_2 as given in equation (4), so that,

$$f(x) = g^2 P_2 = -3\beta[s + a(E)]/\alpha\beta_0^4\mathscr{E}^2 \tag{54}$$

where $\beta = (\mathscr{E}^2 - m^2c^4)^{1/2}/\mathscr{E}$, $a(E) = \tau^{-1}(E) + (dA/dE) - (d^2D/dE^2)$, $\mathscr{E} = E + mc^2$, $\beta_0 = \beta(\mathscr{E} = \mathscr{E}_0)$, and s corresponds to the variable t in the Laplace space. For simplicity we approximate $a(E)$ as its average value between E_0 and E, $\overline{a(E)} = [a(E_0) + a(E)]/2$, in such a way that by substitution of equation (54) into equation (A5), for the variable E, we obtain

$$S_2(E, s) = \pm 1.804 \times 10^{-2} \frac{\alpha^{1/2}[1 - (mc^2/\mathscr{E})^2]}{[s + \overline{a(E)}]^{1/2}} \int_{E_0}^{E} F_2(E')dE' \tag{55}$$

where

$$F_2(E) = \frac{52\mathscr{E}^{3/2}}{(\mathscr{E}^2 - m^2c^4)^{5/4}} - \frac{\mathscr{E}^{7/2}}{(\mathscr{E}^2 - m^2c^4)^{9/4}} - \frac{93}{\mathscr{E}^{1/2}(\mathscr{E}^2 - m^2c^4)^{1/4}},$$

and $a(E) = \tau^{-1} + (\alpha/3)(\beta^{-1} + 3\beta - 2\beta^3)$. To determine $S_2(E, t)$ we apply the inverse Laplace transform $\mathscr{L}^{-1}[(s + \bar{a})^{-1/2}] = (\pi t)^{-1/2} \exp(-\bar{a}t)$:

$$S_2(E, t) = \pm 1.018 \times 10^{-2}(\alpha/t)^{1/2}[1 - (mc^2/\mathscr{E}_0)^2] \exp(-\bar{a}t) \int_{E_0}^{E} F_2(E')dE' \tag{56}$$

for the steady state case the corresponding S_2 value is obtained by putting $s = 0$ ($t \to \infty$) in equation (55),

$$S_2(E) = \pm 1.804 \times 10^{-2}(\alpha/\bar{a})^{1/2}[1 - (mc^2/\mathscr{E}_0)^2] \int_{E_0}^{E} F_2(E')dE' \tag{57}$$

both expressions, $S_2(E, t)$ and $S_2(E)$ are very sensitive to the acceleration parameters α, τ, whose values depend on the astrophysical sources; in the case of solar flare sources, these parameters have not yet been determined with precision. For the coronal plasma of flares, the more usual values (e.g., Miller 1991) fall in the ranges $0.005 \leq \alpha \leq 1\ \mathrm{s}^{-1}$ and $0.05 \leq \tau \leq 1$ s. Similarly, the values of E_0 are not very well defined; nevertheless, some bounds can be settled on the basis of particle energy losses prevailing at the sources: usual values are $E_0 \sim 1$ MeV per nuclear for ions, and $E_0 \sim 1$ keV for electrons. The evaluations of $|S_2(E, t)|$ and $|S_2(E)|$ for acceleration by MHD turbulence in Figure 8 show that the relative error is always extremely small for values of α and τ within the mentioned ranges. We conclude that the WKBJ method is a powerful tool to solve transport equations of energetic particles with a high degree of reliability.

2. The derived time-dependent spectra from equation (15) allows the study of the evolution of energetic particles that have been accelerated by plasma turbulence; for acceleration times long enough ($t \rightarrow \infty$), the process reaches an equilibrium state, and, hence, the behavior of the energetic distributions may be described by the steady state spectra derived from equation (20). Such a tendency toward equilibrium is due to the fact that the acceleration process is associated to the plasma turbulence relaxation. Consequently, the time for the acceleration process to reach equilibrium depends on the specific type of turbulence, the plasma parameters, and the initial energetic distribution of particles going into the process. Miller et al. (1990) determined numerically the equilibrium times for stochastic Fermi acceleration of protons that are continuously injected with a monoenergetic energy of 1 MeV. We have reproduced analytically these results in Figures (3) and (4). In Figures (5) and (6) we show results of equilibrium times, for the same process and same acceleration parameters α, τ, E_0, but with an initial energy distribution proceeding from preacceleration in a MNCS. Figure (5) shows that equilibrium times are slightly longer than with monoenergetic injection; however, such a discrepancy decreases as the particle energy increases, so that for $E > 10^4$ MeV the equilibrium times are basically the same. The corresponding analysis for acceleration by Langmuir turbulence with monoenergetic injection (Figs. 7*a* and 7*b*) indicates that the equilibrium times are notoriously longer than the corresponding times with acceleration by MHD turbulence. Also, it can be appreciated in those figures that equilibrium times increase rapidly when the acceleration efficiency parameter K_L decreases; that is, when the conditions of density and temperature tend toward the high solar corona (lower density and higher temperature) and conversely. We conclude that equilibrium times are sensitive to the kind of accelerating turbulence but only slightly sensitive to the kind of involved particle injection; that is, a very simple injection, such as the monoenergetic one, gives a quite similar description of the tendency toward the equilibrium as an energy-dependent injection spectrum.

3. Particle disappearance from a stochastic acceleration process may be of different nature, diffusive escape, convective escape, fragmentation, catastrophic escape, etc. (e.g., Dröge & Schlickeiser 1986). The escape process may or may not be dependent on the particle parameters, energy, magnetic rigidity, or velocity. For instance, in the case of a catastrophic opening of a closed magnetic source topology, particles of all kinds are ejected simultaneously, but if particles escape by spatial diffusion, there is a dependence on particle velocity and on the parallel diffusion coefficient (which in turn depends on the turbulence spectrum). We have considered velocity-dependent escape times ($\tau \sim \beta^{-\mu}$) and computed steady state spectra by introducing τ into the expression of $a(E)$ in equation (20). It was found (Fig. 9) that for $\mu > 0$ the spectra for $E \geq mc^2$ tend rapidly toward a power-law; however, the main difference with respect to the spectra with $\tau =$ constant ($\mu = 0$) is of qualitative rather than quantitative nature. We conclude that, in first approximation, evaluations of particle spectra with $\tau =$ constant give a reliable description of the actual particle energy spectra in steady state conditions.

APPENDIX A

THE WKBJ METHOD

The theory known as WKBJ (Wentzel, Kramer, Brillouin, & Jeffreys) is a useful tool to derive global approximations to the solution of linear differential equations of any order. In its formal treatment (e.g., Bender & Orszag 1978), the solution of a linear differential equation is represented by

$$y(x) \sim \exp\left[\sum_{j=0}^{\infty} \Delta^{j-1} S_j(x)\right]. \tag{A1}$$

In order to obtain an approximated solution of a given equation of the type

$$\varepsilon^2 \frac{d^2 y}{dx^2} - f(x)y = 0 \qquad [f(x) > 0] \tag{A2}$$

we substitute equation (A1) into equation (A2) obtaining the following sequence of equations

$$\dot{S}_0^2 = f(x)\,, \qquad 2\dot{S}_0 \dot{S}_1 + \ddot{S}_0 = 0\,, \qquad 2\dot{S}_0 \dot{S}_j + \ddot{S}_{j-1} + \sum_{r=1}^{j-1} \dot{S}_r \dot{S}_{j-1} = 0 \qquad (j \geq 2) \tag{A3}$$

where the dots indicate derivatives with respect to x. Solving this set of coupled equations, and by substitution of the solutions into equation (A1) we obtain the solution of equation (A2) by the WKBJ theory in its more general form.

If the solution of equation (A2) is built using only the first term of the equation sequence (A3), the so-called *geometrical optics approximation* (e.g., Bender & Orszag 1978) is obtained.

Solving simultaneously the two first equations of the sequence (A3), we obtain S_0 and S_1, giving the so-called approximation of the *physical optics*, which provides a more representative description of the exact solution $y(x)$.

At this level the obtained approximated solution corresponds to the so-called *Liouville-Green transformation* (Olver 1974). The solution of equation (A2), is then obtained in terms of the *physical optics approximation* (or equivalent the Liouville-Green transformation) as

$$y_{op}(x) = c_1 f(x)^{-1/4} \exp\left[-\int_a^x f(x')^{1/2}dx'\right] + c_2 f(x)^{-1/4} \exp\left[\int_a^x f(x')^{1/2}dx'\right], \tag{A4}$$

where c_1 and c_2 are constants, and a is the initial value of the variable x. Now the relative error between the approximation of the *physical optics*, equation (A4), and the exact solution $y(x)$ is given by $\Delta S_2(x)$ (Bender & Orszag 1978). $S_2(x)$ is obtained by solving simultaneously the first three equations of sequence (A3):

$$S_2(x) = \pm \int_a^x \left\{\frac{\ddot{f}(x')}{8[f(x')]^{3/2}} - \frac{5[\dot{f}(x')]^2}{32[f(x')]^{5/2}}\right\}dx'; \tag{A5}$$

hence equation (A5) gives the relative error that is carried out when only the two first terms of the WKBJ series are used to solve a linear differential equation.

APPENDIX B

SOLUTION OF THE TRANSPORT EQUATION IN THE ULTRARELATIVISTIC RANGE ($\beta \cong 1$) FOR MHD TURBULENCE, WITH $\tau = cst$ AND $D(p) \sim p^2/\beta$

B1. TIME-DEPENDENT SOLUTION

The solution to equation (3) of the text, for the particular case of ultrarelativistic energies, is obtained by applying the Laplace transform to the variable t. Since in this case $\beta \cong 1$, equations (31)–(32) reduce to $A(E) = AE$ and $D(E) = DE^2$, where A and D are constants; hence, applying the transform and the variable change $\ln(E/E_0) = z$, equation (3) reduces to a differential equation with constant coefficients, namely,

$$\frac{d^2\tilde{N}(z, s)}{dz^2} + \left(\frac{4D - A}{D}\right)\frac{d\tilde{N}(z, s)}{dz} + \left[\frac{2D - A - (s + \tau)^{-1}}{D}\right]\tilde{N}(z, s) = -\frac{[\tilde{Q} + N(0)]}{D}, \tag{B1}$$

where $\tilde{N}(z, s)$ denotes the Laplace transform of $N(z, t)$.

The solution of the homogeneous part of equation (B1) in terms of the original variable is $\tilde{N}_H = C_1\tilde{N}_1 + C_2\tilde{N}_2$, where $\tilde{N}_1 = (E/E_0)^{\lambda 1}$ and $\tilde{N}_2 = (E/E_0)^{\lambda 2}$, with C_1 and C_2 constants, E_0 an initial energy, and

$$\lambda_{1,2} = -\frac{4D - A}{2D} \mp D^{-1/2}\left[\frac{(4D - A)^2}{4D} + A - 2D + \tau^{-1} + s\right]^{1/2}. \tag{B2}$$

To build the particular solution, the method of Green functions is used (e.g., Arfken 1970):

$$\tilde{N}_p(E, s) = -\frac{1}{DE_0}\int_{E_0}^E [\tilde{N}(E', 0) + \tilde{Q}(E', s)]G(E, E', s)dE', \tag{B3}$$

where the Green function in this case is

$$G(E, E, s) = -(E/E')^{\lambda_1}/(\lambda_2 - \lambda_1) \qquad (E \geq E' \geq E_0), \tag{B4}$$

so that the particular solution may be expressed in the following form:

$$N_p(E, t) = \frac{1}{DE_0}\int_{E_0}^E \mathscr{L}^{-1}\left[\frac{(E/E')^{\lambda_1}}{(\lambda_2 - \lambda_1)}\right]N(E', 0)dE' + \frac{1}{DE_0}\int_{E_0}^E \mathscr{L}^{-1}\left[\frac{(E/E')^{\lambda_1}}{s(\lambda_2 - \lambda_1)}\right]q(E')dE' \quad \text{(particles per energy unit)}. \tag{B5}$$

Using the inverse transform equation (B5) becomes

$$N_p(E, t) = \frac{1}{(4\pi Dt)^{1/2}E_0}\exp\left\{-\left[\frac{(4D - A)^2}{4D} + \frac{1}{\tau} + A - 2D\right]t\right\}\int_{E_0}^E N(E', 0)\left(\frac{E}{E'}\right)^{-(4D-A)/A}$$
$$\times \exp\left\{-\frac{[\ln(E/E')]^2}{4Dt}\right\}dE' + \frac{1}{(4\pi D)^{1/2}E_0}\int_{E_0}^E q(E')\left(\frac{E}{E'}\right)^{-(4D-A/2D)} dE'$$
$$\times \int_0^t t'^{-1/2}\exp\left\{-\left[\frac{(4D - A)^2}{4D} + \frac{1}{\tau} + A - 2D\right]t' - \frac{[\ln(E/E')]^2}{4Dt'}\right\}dt' \quad \text{(particles per energy unit)}. \tag{B6}$$

A similar expression was previously obtained by Melrose (1980 [vol. 2, p. 114]), in terms of the time integral, but it was not solved and is not dimensionally consistent. The time integral in the second term of the right-hand side of equation (B6) is not a direct one; to effectuate it, let us denote the integral in the time by $I(t)$, so

$$I(t) = \int_0^t t'^{-1/2}\exp[-\mathscr{X}^2 t' - \mathscr{Y}^2/t']dt' \tag{B7}$$

with the corresponding identifications for $\mathscr{X}$ and $\mathscr{Y}$. Using the variable change $t = x^2$, the asymptotic properties of the error function and comparing with the asymptotic integral for $t \to \infty$ (Abramowitz & Stegun 1968) it is found that

$$I(t) = (\pi^{1/2}/2\mathscr{X})\{[\mathrm{erf}\,(Z_1) - 1]\exp(2\mathscr{X}\mathscr{Y}) + [\mathrm{erf}\,(Z_2) + 1]\exp(-2\mathscr{X}\mathscr{Y})\,, \tag{B8}$$

where $Z_{1,2} = \mathscr{X}t^{1/2} \pm \mathscr{Y}t^{-1/2}$. Introducing equation (B8) into equation (B6) it is obtained that

$$N_p(E, t) = \frac{1}{(4\pi D)^{1/2}E_0}\int_{E_0}^{E}\left(\frac{E}{E'}\right)^{-4D-A/2D}\left[\frac{N(E', 0)}{t^{1/2}}\exp\left(\frac{-a_g t - b_g}{t}\right) + \frac{q(E')}{2}\left(\frac{\pi}{a_g}\right)^{1/2}\right.$$
$$\left.\times\{[\mathrm{erf}\,(Z_1) - 1]\exp(M_1) + [\mathrm{erf}\,(Z_2) + 1]\exp(-M_1)\}\right]dE' \quad \text{(particles per energy unit)}\,. \tag{B9}$$

where

$$a_g = (1/\tau) + (A - 2D) + (4D - A)^2/4D\,, \qquad b_g = (1/4D)[\ln(E/E')]^2\,,$$
$$M_1 = D^{-1/2}[\ln(E/E')][(1/\tau) + (A - 2D) + (4D - A)^2/4D]^{1/2}\,,$$

and

$$Z_{1,2} = (a_g t)^{1/2} \pm \ln(E/E')/2(Dt)^{1/2}\,.$$

Now, as discussed in § 2, the Green function given in equations (B4) satisfies completely the boundary conditions [when $E \to \infty$, $G(E, E', s)$ is a finite decreasing function and it takes a constant value when $E \to E_0$, as required by cosmic particle spectra], so that the two homogeneous solutions $\tilde{N}_1$ and $\tilde{N}_2$ are not useful within this context. Therefore, the general solution of equation (B1) is given by the particular solution, so that for the case of MHD turbulence with $A = 4D = (4/3)\alpha$ (e.g., Melrose 1980), equation (B9) may be rewritten as

$$N(E, t) = \frac{1}{(4\pi\alpha/3)^{1/2}E_0}\int_{E_0}^{E}\left[\frac{q_{t=0}(E')}{t^{1/2}}\exp\left[\frac{-a_g t - b_g}{t}\right] + \frac{q(E')}{2}\left(\frac{\pi}{a_g}\right)^{1/2}\right.$$
$$\left.\times\{[\mathrm{erf}\,(Z_1) - 1]\exp(M_1) + [\mathrm{erf}\,(Z_2) + 1]\exp(-M_1)\}\right]dE' \quad \text{(particles per energy unit)}\,. \tag{B10}$$

Here $q_{t=0}(E') = N(E', 0)$, $a_g = (\alpha/3)(2 + 3/\alpha\tau)$, $b_g = [\ln(E/E')]^2/(4\alpha/3)$, $M_1 = (3a_g/\alpha)^{1/2}\ln(E/E')$ and $Z_{1,2} = (a_g t)^{1/2} \pm \ln(E/E')/(4\alpha t/3)^{1/2}$. In the specific case of monoenergetic injection, $q(E) = \mathscr{N}\delta(E - E)$, we obtain the following analytical expression:

$$N(E, t) = \frac{1}{(4\pi\alpha/3)^{1/2}E_0}[\![(N_0/t^{1/2})\exp(-a_g t - b_g/t) + (q_0/2)(\pi/a_g)^{1/2}\{[\mathrm{erf}\,(Z_1) - 1]$$
$$\times\exp(M_2) + [\mathrm{erf}\,(Z_2) + 1]\exp(-M_2)\}]\!] \quad \text{(particles per energy unit)}\,, \tag{B11}$$

where $\mathscr{N}$ in the injection spectrum has been denoted by q_0 (particles s^{-1}) indicating continuous injection, and has been denoted by N_0 to indicate the total number of impulsively injected particles in a pulse at $t = 0$, and M_2 is the same as M_1 evaluated at $E' = E_0$. It should be emphasized that this particular case has been previously analyzed by Kardashev (1962).

B2. STEADY STATE SOLUTION

Similarly to the time-dependent case when $\beta \approx 1$ and $A = 4D = (4\alpha/3)$, for the steady state situation ($s = 0$) the following equation is obtained:

$$N(E) = \frac{1}{(2/3)\alpha E_0(2 + 3/\alpha\tau)^{1/2}}\int_{E_0}^{E} q(E')\left(\frac{E}{E'}\right)^{-(2+3/\alpha\tau)^{1/2}} dE' \quad \text{(particles per energy unit)}\,. \tag{B12}$$

It can be seen that equation (B12) can be directly obtained from equation (B10) when $t \to \infty$, in which case the first term of equation (B10) goes to zero and erf $(Z_1) \to 1$. In the specific case of monoenergetic injection the previous equation becomes an analytical power law of the type

$$N(E) = (q_0/2)[(\alpha/3)(2 + 3/\alpha\tau)^{1/2}E_0]^{-1}(E/E_0)^{-(2+3/\alpha\tau)^{1/2}} \quad \text{(particles per energy unit)}\,. \tag{B13}$$

APPENDIX C

EVALUATION OF THE ERROR CONTROL FUNCTION FOR THE STEADY STATE CASE

Another alternative to quantify the validity of the WKBJ approximation in the context of an evolution equation is by applying the Liouville transformation (Olver 1974) to the normal form of the equation. In the present case, we apply it to equations (6) and (17) for the time-dependent and steady state situations, respectively. In both cases the transformation leads to a differential equation of the form

$$\frac{d^2\mathscr{U}}{d\xi^2} = (1 + \Phi)\mathscr{U}\,, \tag{C1}$$

where $\mathscr{U} = [\xi'(\eta, s)]^{1/2}\tilde{N}(\eta, s)$, and $\xi(\eta, s) = \int_{\eta_0}^{\eta_{1/2}} (\eta', s)d\eta'$, with $s = 0$ in the steady state case. By solving equation (C1) with $\Phi = 0$ the Liouville-Green approximation to the solution of the homogeneous part of equations (6) or (17) (Olver 1974) is obtained which coincides with the approximation of the "*optical physics*" within the WKBJ theory (see Appendix A). In these circumstances the integrated value of Φ designated as "*error control function*" (F) (Olver 1974) is a direct measure of the information that is lost by applying the Liouville-Green approximation. In the particular case of Fermi-type acceleration by MHD turbulence the value of F in terms of the variable energy E, in the steady state case, is given as

$$F(E) = \int_{E_0}^{E} [(4R\ddot{R} - 5\dot{R}^2)/16R^3](\beta_0/\beta)^2 dE' \tag{C2}$$

where

$$R = (l_1 \beta + l_2)/\mathscr{E}^2 , \qquad l_1 = [(3/\alpha\tau) + (3\beta - 2\beta^3)]/\beta_0^4 , \qquad l_2 = -\beta_0^{-4} ,$$

$$\dot{R} = -[2l_2 \beta\mathscr{E}^2 + l_1(mc^2)^2 + 2l_1 \beta^2\mathscr{E}^2 - 3\beta(mc^2)^2(1 - 2\beta^2)/\beta_0^4]/\beta\mathscr{E}^5 ,$$

$$\ddot{R} = \frac{6(l_2 + l_1 \beta)}{\mathscr{E}^4} - \frac{(mc^2)^4\{[3(1 - 2\beta^2)/\beta_0^4] + l_1/\beta\}}{\beta^2\mathscr{E}^8} + \frac{7(mc^2)^2[-3(1 - 2\beta^2)/\beta_0^4 + l_1/\beta]}{\mathscr{E}^6} + \frac{12(mc^2)^4}{\beta_0^4\mathscr{E}^8} .$$

REFERENCES

Abramowitz, M., & Stegun, I. A. 1965, Handbook of Mathematical Functions (New York: Dover)
Achterberg, A. 1981, A&A, 97, 259
Arfken, G. 1970, Mathematical Methods for Physicists (New York: Academic)
Barbosa, D. D. 1979, ApJ, 233, 383
Bender, C. M., & Orszag, S. A. 1978, Advanced Mathematical Methods for Scientists and Engineers (New York: McGraw-Hill)
Borowsky, J. E., & Eilek, J. A. 1986, ApJ, 308, 929
Braginskii, S. I. 1965, Rev. Plasma Phys., 188, 205
Bykov, A. M. 1992, in Particle Acceleration in Cosmic Plasmas (AIP Conf. Proc. 264), ed. G. P. Zank & T. K. Gaisser (New York: AIP), 221
Dröge, W. 1989, in Cosmic Gamma Rays, Neutrinos, and Related Astrophysics, ed. M. M. Shapiro & J. P. Wefel (Dordrecht: Kluwer), 237
Dröge, W., & Schlickeiser, R. 1986, ApJ, 305, 909
Dung, R., & Schlickeiser, R. 1990, A&A, 237, 504
Forman, M. A., & Webb, G. M. 1985, in Collisionless Shocks in the Heliosphere, ed. B. T. Tsurutani & R. G. Stone (Washington: AGU), 91
Gallegos-Cruz, A., & Pérez-Peraza, J. 1990, Proc. Soc. Nuclear Mex., I, 294
Gallegos-Cruz, A., Pérez-Peraza, J., Miroshlichenko, L. I., & Vashenyuk, E. V. 1993, Adv. Space Res., 13, 187
Ginzburg, V. L. 1958, in Prog. Elementary Particle Cosmic-Ray Phys., 4, 339
Ginzburg, V. L., & Sirovatskii, S. I. 1964, The Origin of the Cosmic Rays (Oxford: Pergamon)
Hasselmann, K., & Wibberenz, G. 1968, Geophys., 34, 353
Jokipii, J. R. 1966, ApJ, 146, 480
———. 1977, Proc. 15th Internat. Cosmic Ray Conf. (Plovdiv), 1, 429
Kaplan, S. A. 1956, Soviet Phys.—JETP 2, 203
Kaplan, S. A., & Tsytovich, V. N. 1973, Plasma Astrophysics (Oxford: Pergamon)
Kardashev, N. S. 1962, Soviet Astron.-AJ, 6(2), 317
Kulsrud, R. M., & Ferrari, A. 1971, Ap. Space Sci., 12, 302
Lerche, I., & Schlickeiser, R. 1985, Proc. 19th Internat. Cosmic Ray Conf. (La Jolla), 3, 222
Mandzhavidze, N., Ramaty, R., Akimov, V. V., & Leikov, N. G. 1993, Proc. 22d Internat. Cosmic Ray Conf. (Dublin), 3, 119
Mathews, J., & Walker, R. L. 1973, Mathematical Methods of Physics (Reading: Addison Wesley), 27
Melrose, D. B. 1976, MNRAS, 176, 15
———. 1980, Plasma Astrophysics (New York: Gordon & Breach)
Melrose, D. B. 1986, Instabilities in Space and Laboratory Plasmas (Cambridge: Cambridge Univ. Press), 233
Miller, J. A. 1991, ApJ, 376, 342
Miller, J. A., Guessoum, N., & Ramaty, R. 1990, ApJ, 361, 701
Miller, J. A., & Ramaty, R. 1987, Sol. Phys., 113, 195
Miller, J. A., Ramaty, R., & Murphy, R. J. 1987, Proc. 20th Internat. Cosmic Ray Conf. (Moscow), 3, 33
Mullan, D. J. 1980, ApJ, 237, 244
Olver, F. W. J. 1974, Asymptotic and Special Functions (New York: Academic), 193
Parker, E. N., & Tidman, D. A. 1958, Phys. Rev., 11, 1206
Pérez-Peraza, J., Gálvez, M., & Lara, R. 1977, Proc. 15th Internat. Cosmic Ray Conf. (Plovdiv), 5, 23
———. 1978, Adv. Space Res., 18, 365
Pérez-Peraza, J., & Gallegos-Cruz, A. 1994a, Geof. Int., in press
———. 1994b, ApJS, 90, 669
Press, W. H., Brian, P. F., Teukolsky, S. A., & Vetterling, W. T. 1986, Numerical Recipes (Cambridge: Cambridge Univ. Press)
Priest, E. R. 1973, ApJ, 181, 227
———. 1982, Solar Magnetohydrodynamics (Dordrecht: Reidel)
Ramaty, R. 1979, in Particle Acceleration Mechanisms in Astrophysics, (AIP Conf. Proc. 56), ed. J. Arons, C. Max, & C. McKee (New York: AIP), 135
Rose, W. K., Beall, J. H., Guillory, J., & Kaimer, S. 1987, ApJ, 314, 95
Rosenbluth, M. N., MacDonald, W. M., & Judd, D. L. 1957, Phys. Rev., 107, 1
Schatzman, E. 1966, in Summer School of Theoret. Phys. Les Houches (New York: Gordon & Breach), 229
Schlickeiser, R. 1984, A&A, 136, 227
———. 1989, ApJ, 336, 243
———. 1994, ApJS, 90, 929
Smith, D. F., & Breach, S. H. 1989, ApJ, 344, 1004
Steinacker, J., & Miller, J. A. 1992, in Particle Acceleration in Cosmic Plasmas, (AIP Conf. Proc. 264), ed. G. P. Zank & T. K. Gaisser (New York: AIP), 235
Steinacker, J., & Schlickeiser, R. 1989, A&A, 224, 229
Syrovatskii, L. I. 1961, Sov. Phys.-JETP, 13, 1257
Tsytovich, V. N. 1977, Theory of Turbulent Plasma (New York: Consultants Bureau)
Tverskoi, B. A. 1967, Soviet Phys.-JETP, Vol. 25, 317
Wang, Y. M., & Schlickeiser, R. 1987, ApJ, 313, 200