Geofísica Internacional (1993), Vol. 33, Num. 2, pp. 311-331

The role of fluctuational acceleration in the generation of solar particles

J. Pérez-Peraza[1] and A. Gallegos-Cruz[2]
[1]*Instituto de Geofísica, UNAM, México, D. F. MEXICO*
[2]*I.N.A.O.E., Tonantzintla, Puebla, MEXICO*

Received: October 7, 1992; accepted:February 19, 1993.

RESUMEN

La aceleración estocástica de partículas es fundamentalmente un proceso de difusión en el espacio fase de energía. A pesar del comportamiento estadístico del proceso de difusión, se establece una tendencia promedio de ganancia de energía de carácter determinístico, que es usualmente designada como aceleración sistemática. En el caso particular de la Física de Rayos Cósmicos, a menudo se considera solamente la tasa de aceleración sistemática, ignorándose los efectos de dispersión en el espacio de energía, los cuales son identificados como una tasa fluctuacional de aceleración. Sin embargo, ha sido demostrado por varios autores que, dependiendo de las propiedades de la turbulencia involucrada en la aceleración, la tasa de aceleración sistemática puede ser ineficiente en algunos casos, e incluso ser nulificada por procesos competitivos de pérdida de energía, de tal forma que la producción de partículas energéticas es debida exclusivamente a los efectos de dispersión en energía. En este trabajo se calcula separadamente la contribución de ambas tasas de cambio de energía, para evaluar su importancia en la producción de partículas energéticas. Se consideran dos mecanismos de aceleración, el proceso Fermi y la aceleración resonante por turbulencia magnetosónica (cuando el armónico S=0). Se resuelve analíticamente la ecuación de transporte para el caso estacionario y para el caso dependiente del tiempo. Se encuentra que la contribución de los efectos de dispersión en energía al espectro de partículas al nivel de sus fuentes no puede ser considerada como una mera fluctuación en el flujo de partículas, sino que representa en algunos casos una importante sobre-producción de partículas y en otros casos una importante depresión de partículas en el espectro de aceleración. La relevancia de esos efectos de difusión en energía depende principalmente de la eficiencia del proceso acelerador, la correlación entre la población inicial de partículas con el espectro de velocidades de la turbulencia, y de la proporción relativa entre las diferentes clases de interacciones de las partículas con los agentes aceleradores. Se establecen los límites bajo los cuales los efectos de difusión en energía podrían ser ignorados relativos a la tasa de aceleración sistemática. Se concluye que, con algunas excepciones, en el caso estacionario el espectro de energía derivado exclusivamente con base en la tasa sistemática no describe el flujo real de las partículas aceleradas, y que el apelativo de tasa fluctuacional de aceleración no es apropiado. El cálculo de flujo de radiación secundaria con base al espectro de las partículas aceleradas debe de tomar en consideración la limitante de considerar únicamente la tasa de aceleración sistemática.

PALABRAS CLAVE: Física de Rayos Cósmicos, aceleración estocástica, tazas sistemáticas y difusivas de aceleración.

ABSTRACT

Stochastic particle acceleration is essentially a diffusion process in energy phase space. In spite of the statistical behavior of the diffusion process, there is an average energy gain tendency of deterministic nature which is usually called Systematic Acceleration. In practice only the systematic acceleration rate has been considered, ignoring effects of the diffusion process, usually identified as a fluctuational acceleration rate. However, depending on the nature of the phase velocity spectrum of the turbulence, or on competitive energy loss processes, the average systematic acceleration rate may become inefficient and even null, so that energetic particle production is due only to the energy spread effects (diffusion in energy). We calculate separately the contribution of both energy change rates, in order to evaluate the importance of particle production by each. We consider the classical Fermi process and turbulent acceleration by magnetosonic waves (for the case in which S=0 in the resonance condition). The transport equation is solved analytically for the steady state and for the time-dependent situation. We find that the contribution of fluctuational acceleration to the source solar particle spectrum cannot be considered as mere particle flux fluctuations, but may represent an important overproduction in some cases and particle depression in others. The relevance of energy spread effects is related to the efficiency of the energy gain: the nature of the initial particle population relative to the velocity spectra of turbulence, and thus the relative proportion among the different kind of interactions of particles with accelerating agents. We discuss the conditions under which diffusion in energy effects should be ignored relative to the average energy gain rate. With some exceptions in the stationary case, energy spectra derived on the basis of systematic acceleration alone cannot describe the real particle flux. This must be taken into account in calculations of the flux of secondary radiation.

KEY WORDS: Cosmic ray physics, stochastic acceleration, systematic and diffusive acceleration rates.

1. INTRODUCTION

Energetic particle motion in generating sources is defined by electric charge and momentum and controlled by the local em fields. The fields are composed of an average field, on some scale, plus a turbulent component on a smaller scale, associated with waves in the plasma. Turbulent fields cause random scattering of particles from the paths they would have taken in the mean em fields alone. Elementary interactions of particles with turbulent fields may result in a change in particle energy. This stochastic effect on the energetic particle distribution has been studied by means of continuity equations, usually called transport equations, and in particular the diffusion-convection equation. There are several ways to derive this equation in order to study the evolution of the energy distribution of the accelerated particles (e.g. Jones, 1991). The most common methods are derived from the collisionless Boltzmann transport equation and from the Chapman-Kolmogorov equation. In the former a quasi-linear formalism is established, by assuming that the particle distribution function has an average part plus a smaller fluctuating part. In the second method, expected values and mean square deviations of the particle momentum along the unperturbed path of one particle are usually calculated. Both formalisms may lead to a generalized Chandrasekhar equation which, once adapted to the specific case of the generation of energetic particles, becomes a Fokker-Planck type equation. One way to do so, within the frame of the Boltzmann-Vlasov formalism, is from the momentum diffusion equation

$$\frac{\partial f}{\partial t} + \nabla . S + \frac{1}{p^2}\left(p^2 J_p\right) = 0 \qquad (1)$$

which is usually derived from the transport equation in phase space

$$\frac{\partial f}{\partial t} = \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2 D(p)\frac{\partial f}{\partial p}\right] \qquad (2)$$

where the divergence of the streaming in space is assumed to vanish ($\nabla . S=0$), and the adiabatic compression and particle continuous energy losses are ignored within the streaming in momentum space J_p (e.g. Forman and Webb, 1985).

Equation (1) is valid for isotropic plasma turbulence and assumes that the particle phase-space density f(p,r,t) is spatially homogeneous and isotropic. The momentum-diffusion coefficient $D(p)=<\Delta p>^2/\Delta t$ describes the effect of plasma turbulence on particles; Dp is the small change in momentum undergone by a particle during each random interaction with the turbulence. In such an isotropic particle distribution the number density N(E,t) may thus be related to the phase-space density by $N=4\pi p^2 f(dp/dE)$, so that (1) may be expressed in the form of a Fokker-Planck equation (e.g. Tsytovich, 1977),

$$\frac{\partial N(E,t)}{\partial t} = -\frac{\partial}{\partial E}\left[\left\langle\frac{dE}{dt}\right\rangle N(E,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial E^2}\left[\left\langle\frac{dE^2}{dt}\right\rangle N(E,t)\right] \qquad (3)$$

where the first Fokker-Planck coefficient describes the average energy change, as a kind of systematic energy change rate

$$\left\langle\frac{dE}{dt}\right\rangle = \frac{1}{p^2}\frac{\partial}{\partial p}\left[vp^2 D(p)\right] \qquad (4)$$

and the second Fokker-Planck coefficient is the diffusion in energy space around the average energy change rate,

$$\left\langle\frac{dE^2}{dt}\right\rangle = 2v^2 D(p) \qquad (5)$$

The meaning of these two coefficients is the following: particles gain and lose energy through the stochastic elementary interactions with turbulent agents moving with average velocity higher than the characteristic velocity of particles, but on the average there is a net energy gain. In spite of the statistical (rather than secular) behavior of the process, this net energy gain may be seen as a unidirectional systematic energy increase at the average rate <dE/dt>. Because of the stochastic nature of the elementary accelerating interactions, the energy change ΔE is not necessarily the same for particles of the same energy E: some of them may even decrease their energy in a given interaction, but there is a spread in the energy change around the average, which results in a dispersion of energy for particles of the same energy after each elementary interaction. This dispersion is determined by the variance of the particle energy at the rate of (dE^2/dt).

Back to eq. (3), in cosmic ray physics several terms are added depending on the specific problems to be solved (e.g. Ginzburg, 1958). In the case of source particle acceleration we usually add the particle injection into the acceleration process Q(E,t), the probability of particle disappearance $1/\tau$ from the acceleration volume by escape or nuclear transformations, and a rate of deceleration by different processes of energy losses (-dE/dt) (usually considered together with the term of systematic acceleration). As a first approximation,we neglect this last factor in this paper:

$$\frac{\partial N(E,t)}{\partial t} = -\frac{\partial}{\partial E}\left[\left\langle\frac{dE}{dt}\right\rangle N(E,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial E^2}\left[\left\langle\frac{dE^2}{dt}\right\rangle N(E,t)\right] \frac{N(E,t)}{\tau(E,t)} + Q(E,t) \qquad (6)$$

where the second term of the right side represents diffusion in energy space. The importance of this term has been discussed in connection with suprathermal particle generation (e.g. Davis, 1956; Ginzburg and Syrovatskii, 1964). Its meaning derives from the fact that, even when the average energy gain is zero, energy diffusion produces particle acceleration. Schatzman (1966) derived the flux of particles produced by the diffusion term in equation (3), when the average energy loss within J_p in eq. (2) equals the average energy gain, so that the first term in eq. (3) becomes zero. In the particular case of ion sound wave turbulence, it has been argued that the acceleration of non-relativistic particles is determined only by energy diffusion (e.g., Lacombe and Manganey, 1969). In the case of Fermi acceleration, Ginzburg and Syrovatskii (1964) and Melrose (1980) have pointed out that the ratio of the numerical coefficients in

(4) and (5) is usually in the order of 4:1, though obviously it changes from one mechanism to another. However, this ratio relates only to the numerical coefficients of the rates $<dE/dt>$ and $<dE^2/dt>$ and not to the ratio between the two complete terms of the right side of (3). For an evaluation of the effect of these terms on the density of the accelerated particles and their energy distribution, equation (3) or (6) needs to be solved. Therefore, the relative effect of both terms in eq. (3) could vary greatly depending on the assumptions on the acceleration mechanism, the accelerating turbulence involved and the initial assumptions (mainly the initial particle population).

Depending on the acceleration efficiency of the process, the deviations of the parameters from their average value may be considered to some extent as fluctuations about their characteristic values, which may result in fluctuations of the acceleration parameters. For instance, fluctuations of the mean magnetic field strength and/or matter density in the source entails fluctuations of the hydromagnetic velocity and of the power spectrum of the associated turbulence. Since the em fields are randomly distributed there may be significant deviations of the characteristic average value of the mean free path of interaction of particles with the accelerating agents, and considerable resulting deviations in the permanence of particles in the acceleration volume with respect to the characteristic mean confinement time. All these deviations may result in fluctuations of the characteristic acceleration efficiency, which in turn entails fluctuations around the average acceleration rate. To some extent these fluctuations are included in the effect of the energy diffusion rate (dE^2/dt). However, the effect of these deviations on the energy spectrum of the accelerated particles, which is derived by solving (3) or (6), does not only result in fluctuations in the number density but may, in some cases, determine the particle production. In fact, while it is often argued that some turbulence may be ineffective in accelerating particles, as only energy diffusion is produced, we shall see that (at least in the two cases considered for solar particle production) diffusion in energy may lead to a significant particle production, which turns out to be in some cases the largest contribution as compared to the systematic acceleration rate.

Until now, the relative importance of the systematic and diffusion rates in the generation of energetic particles has been discussed on qualitative grounds, for the specific case of $\beta = 1$, with a certain tendency to neglect the diffusion rates. However, for any acceleration process, the diffusion term is proportional to the momentum transfer between the particles and the medium. Thus, for very energetic events, such as stellar flares, this effect may play an important role in the formation and regulation of the shape of the particle energy spectrum and the flux magnitude.

In an earlier paper (Gallegos-Cruz and Pérez-Peraza, 1987), it was found that even in the extreme case when $\beta = 1$, the contribution of the energy spread term cannot be seen in terms of simple fluctuations within the frame of energy distributions of particle flux. It is an important energy-dependent effect modulated by several factors. We shall now consider the general case (for any value of β), by solving the complete transport equation in the steady-state and time-dependent cases, and comparing these solutions with the solutions obtained when the energy diffusion term is neglected. The relative importance of the systematic and the diffusive terms is quantitatively evaluated as follows.

Primary particle acceleration in astrophysical sources appears from the non-relativistic background material. Even secondary acceleration stages of local energetic particles, (e. g. in solar sources) may start out with an important population of non-relativistic or trans-relativistic particles, before the particles reach very high relativistic energies. Under the non-ultrarelativistic approach it is difficult to draw direct conclusions about the relative effect of the systematic and diffusive energy change rates on the particle flux generated, because the dependence of the two right-hand terms in eq. (3) on E and β differs from each other. Also, because the initial energetic level of acceleration (i.e., the kind of initial particle population in a given scenario) modulates the relative contribution of (4) and (5) in the particle flux, any quantification in this regard must be considered within the frame of specific scenarios. As an illustration, in section 3.1 we propose two general scenarios for solar particle generation, each with a specific initial particle population. Two different acceleration processes are worked out for both scenarios. The mathematical expressions derived for the particle energy distributions are presented in section 3.2. Results are given in section 4 and discussions and conclusions are provided in sections 5 and 6.

2. THE RATES OF SYSTEMATIC AND DIFFUSIVE ACCELERATION

Charged particles are accelerated via individual behavior when each particle undergoes separately the same mechanism, or via collective behavior when an important number of particles accelerates together (Schatzman, 1966). Since the density of the accelerated particles in the case of individual behavior is relatively low, their mutual interactions are disregarded in the context of the Boltzmann equation (which allows for binary collisions), so that particle acceleration in these cases is studied as a collisionless problem. In the collective behavior, energy is transferable from a set of particles through a given kind of plasma oscillations or shock waves to another group of particles, and the study of particle acceleration is usually simplified by hydromagnetic approaches. The accelerating force via electric fields may be either secular (deterministic process) or random (stochastic process). Particle acceleration in solar sources, such as flares, has been associated with both fast, impulsive and unidirectional acceleration events where particles gain energy systematically, and slow random events where particles gain and lose energy in elementary interactions with random em fields. In the latter case, due to statistical and thermodynamical equilibrium, there is on the average a net energy gain. Since each particle is randomly accelerated, and undergoes small energy changes, the accelerating elementary interactions of individual particles may be treated as independent events. The behavior of the accelerated particle flux is thus adequately represented by a stochastic process.

Stochastic particle acceleration in nature is associated with plasma turbulence. Turbulence has a tendency to relax through energy deposition on the particles by means of energy transfer processes of a statistical nature. To quantify such an energy deposition several processes have been proposed, such as the betatron mechanism (Swann, 1933), the Fermi mechanism (Fermi, 1949), the magnetic pumping mechanism (Berger *et al*, 1958) and others. Turbulence may be generated by a wide range of instability factors, (e.g., microinstabilities in plasma boundaries), or by expanding shock waves during their passage through the plasma, as is the case of solar flares, etc. Among a wide variety of waves that may co-exist in a magnetized plasma, hydromagnetic turbulence is of particular importance in the context of charged particle acceleration. In fact several authors (e.g. Kulsrud and Ferrari, 1971, Achterberg, 1981) have shown that an adequate characterization of the MHD turbulence allows for the derivation of specific acceleration mechanisms within the frame work of a general theory of wave-particle interaction.

The conditions for the occurrence of energy deposition on the particles and the amount of transferred energy vary widely, depending on the nature of particle flux and their energy distribution, the wave spectrum of the turbulence, and the physical conditions of the medium. The three above mentioned mechanisms may be interpreted in terms of particle interactions with small-scale MHD turbulence (Kulsrud and Ferrari, 1971). In particular, the Fermi mechanism has evolved from the original model where magnetic inhomogeneities behaved as magnetic mirrors, o more recent interpretations in terms of resonant wave-particle interactions, e.g., interaction with a weak turbulent field of isotropic magnetosonic wave superposition (Fisk, 1976), or resonant dispersion with a turbulent field of MHD waves (Achterberg, 1979; Eilek 1984, etc). However, the dynamics of statistical particle collisions is more easily characterized by collisions with magnetic irregularities than in a context of wave-particle theory, which strictly speaking involves non-linear processes. Therefore, both to illustrate and distinguish Fermi acceleration from wave-particle interactions and particle collision with magnetic inhomogeneities, we refer to the "classical" Fermi process when we are dealing with interactions where the accelerating agents are as hard spheres with masses much larger than those of the particles (e.g., Parker and Tidman, 1958) provided that the gyroradius of the particles is smaller or equal than the size of the magnetic irregularities. The latter are of the order of the interaction mean free path ($r_g \leq L \cong \Lambda$). This is a relatively inefficient process as compared to acceleration by resonant wave-particle interactions, where even particles of $r_g >> L$ may interact with effective energy interchange if the resonant condition between particles and waves is fulfilled, so that a much higher interaction frequency and thus a higher acceleration efficiency is expected.

2.1. The classical Fermi process

The original Fermi process seems to be very inefficient in the interstellar medium since the acceleration efficiency increases with decreasing size of the accelerating agents ($\alpha \sim 1/L$). At that level, particles are accelerated by colliding with large-scale moving magnetic irregularities (e.g. Unsold, 1951, Melrose, 1980). However, it is well known that high energy particles are produced in solar flares with a high efficiency: typically 10%-50% of the flare energy is transferred to charged particles. In very localized regions of solar flares, it has been shown (Melrose, 1975; Pérez-Peraza, 1975; Schatten *et al*, 1977; Mullan, 1980) that small-scale inhomogeneities make the Fermi process highly efficient. The reason why the classical Fermi mechanism may not operate in solar events is the lack of selectivity in the mass spectrum of accelerated particles (e.g. Eichler, 1979), which is contradicted by the wide observational variety of relative abundances of heavy nuclei and isotopes and proton/electron ratios from one solar event to another. However, the lack of intrinsic selectivity of the acceleration process is a restriction not only on the Fermi process but on other mechanisms as well. Rather than being directly related to the acceleration mechanism, the selectivity of particle abundances might be associated with the physical structure of the source during particle local transport and ejection from the source, or even with the simultaneous energy losses during acceleration while particles interchange charge with the source matter (Pérez-Peraza, 1981, Pérez-Peraza *et al*, 1982).

The average acceleration rate of the Fermi process may be directly obtained from the corresponding diffusion coefficient,

$$D_p = \left(\frac{\zeta}{4}\right) \alpha \frac{p^2}{\beta} \tag{7}$$

where ζ is a parameter related to the probability of a given type of collision between particles and magnetic irregularities. Depending on whether the interaction is head-on or catch-up, and on the size, hardness and shape of the magnetic irregularities, the value of ζ will fall within the range of $0.25 \leq \zeta \leq 2$ (Ginzburg and Syrovatskii, 1964). However, ζ is not necessarily limited to that range (Melrose, 1980). For $\zeta = 4/3$, $D = (\alpha/3)(p^2/\beta)$ as often stated in the literature. Thus, according to eqs. (4) and (7) the average energy gain rate in its simplest form is

$$A(E) = \langle dE/dt \rangle = \alpha_f \beta W = (4/3)\alpha pc \tag{8}$$

with $W = E + mc^2$ = particle total energy, and

$$\alpha_f = \zeta(u^2/lv) = \zeta(1/\tau)(u/v) = (4/3)\,\alpha \tag{9}$$

where t is the acceleration time, u is the random velocity of the magnetic inhomogeneities, τ is the mean flight time between collisions, and l is the mean free path of particles between magnetic mirrors. According to Melrose (1975) and Mullan (1980) $l < 100$ Km in the solar corona, and according to Pérez-Peraza (1975) $l \sim 1$ Km at the chromospheric level.

For the energy diffusion rate, according to eqs. (5) and (7) we have

$$D(E) = <dE^2/dt> = (1/2)\alpha_f \beta^3 W^2 = (2\alpha/3)\,\beta^3 W^2 \qquad (10)$$

2.2 Resonant acceleration by magnetosonic wave turbulence

In a magnetized plasma, such as the solar atmosphere, when two restoring forces (the magnetic pressure and the plasma pressure) act together, magnetosonic waves are produced. Energy transfer between waves and particles takes place when a resonance is established between some of the properties of the waves and the particles: the resonant wave properties are the wavelength λ or the frequency υ and the phase velocity $V\phi$; the resonant properties of the particles are the velocity v and the gyrosynchrotron frequency ω_i. The magnetosonic fast mode is often associated with acceleration of suprathermal particles (e.g. Eilek, 1979; Achterberg, 1981; Melrose, 1993), and the magnetosonic slow mode may be associated with localized heating of thermal particles (e.g. Barnes, 1966; Kaplan and Tsytovich, 1973).

Energy transfer from the turbulence to a number of resonant particles is essentially by means of non-linear Landau damping, the resonant condition may be found, for instance, in Canuto *et al*, 1978, p. 312). For practical purposes, linearized approaches to the Landau damping are often used, in which case the resonant condition for the harmonic S=0 is given by the inequality $|v - V\phi|t < \pi/k = \lambda/2$, i.e., only those particles in a given initial distribution that have not yet traveled a half-wavelength relative to the wave are able to undergo a resonant energy interchange (Cheng 1974, p. 233). The width of the resonance central peak narrows with time. For a given initial population (for instance a Maxwellian one) of charged particles, those with $v_{mp} > V\phi$ (where v_{mp} is the most probably velocity of the distribution) which satisfy the above resonant condition, give energy to the waves, and those with $v_{mp} \le V\phi$, that satisfy the resonant condition, receive energy from the waves by Landau damping. Thus the net energy interchange is strongly dependent on the relative number of resonant particles with $v >\le V\phi$, i.e. on the sign of $(\partial f/\partial v)_{v=v\phi}$ (where f is the initial velocity distribution of charged particles). A net particle energy gain takes place when the wave-particle interactions occur in the energetic region of the particle spectrum where $(\partial f/\partial v) < 0$. For the case of interactions between the fast magnetosonic mode and suprathermal particles an equivalent resonance condition which is often used in the literature (e.g. Achterberg, 1981) is the relation $\omega - s\Omega - k_{\parallel}v_{\parallel} = 0$, (where $k_{\parallel}$ and $v_{\parallel}$ are the wave vector and particle velocity in the direction parallel to the mean magnetic field), so that for S = 0, the resonant condition that must be satisfied is $kV_a - k_{\parallel}v_{\parallel} = 0$, where V_a is the Alfvén velocity. Since it usually happens that $V_a \le V\phi$ and $(k/k_{\parallel}) > 1$, it turns out that only particles with a component $v_{\parallel} > V_a$ are susceptible to acceleration.

By analyzing a typical "diffraction pattern" of resonance (e.g. Cheng 1974, p. 233) it is found that out of the central peak of resonance (delimited around $V\phi$ = the most probable velocity in the wave phase distribution of the Kolmogorov spectrum), there are subsidiary peaks due to particles that have traveled into neighboring half-wavelengths of the wave potential. These particles rapidly become spread out in phase, so that in principle they contribute little on the average to the new particle distribution, once the initial distribution is obliterated. However, it must be kept in mind that resonant particles may include both trapped and non-trapped particles in the wave potential, since the resonance peaks are independent of the initial amplitude of the wave. Resonance and particle trapping in the wave potential wave are unrelated phenomena.

From this discussion it follows that two kinds of wave-particle energy interchanges may be distinguished: a systematic energy interchange corresponding to the central peak of resonance, and a diffusive interchange related to subsidiary resonance peaks, whose particles may diffuse out or into different resonant energetic bands.

Let us derive the systematic and diffusive energy change rates. The corresponding momentum coefficient diffusion is (e.g. Achterberg, 1981),

$$D_p = \zeta \alpha p^2 \qquad (11)$$

so that for $\zeta \sim 1$, according to eq. (4),

$$A(E) = < dE/dt > = \left(\alpha v/p^2\right)\frac{\partial}{\partial p}\left(p^4\right)$$
$$= 4\alpha\beta c p = 1\alpha\beta^2\,W = \alpha_m\beta^2 W \qquad (12)$$

where $\alpha_m = 4\alpha$, $\beta = v/c$ and c = light velocity. Likewise, according to eq.(5) and (11)

$$D(E) = <dE^2/dt> = 2v^2 D_p = 2\alpha\beta^2c^2p^2 = 2\alpha\beta^4 W^2$$

$$= (\alpha_m/2)\cdot \beta^4 W^2 \qquad (13)$$

where the acceleration efficiency α_m is (e.g., Tverskoi, 1967; Tystovich; 1970, Eilek, 1981)

$$\alpha_m = \frac{2\pi^2 V_\phi^2}{B^2 c}\int_{K_m}^{K_M} K'W(K')dK' \qquad (14)$$

W(k) is the turbulent wave spectrum, usually considered as a Kolmogorov spectrum $\sim k^{-\gamma}$, where the index γ represents the type of involved turbulence, B is the strength of the magnetic field, k_m and k_M define the wavelength range of the magnetosonic turbulence, with $k = 2\pi/\lambda$ = the wavenumber vector.

3. SOLUTIONS OF THE TRANSPORT EQUATION FOR SPECIFIC SCENARIOS

Equation (6) is complicated to solve by analytical methods. However, it is always possible to transform this kind of differential equation into an equation which avoids the first derivative in energy. The reduced form may be solved under certain restrictions by the WKBJ method, as shown in Appendix C. Obviously, the solution depends on the features of the problem, in this case the generation of energetic particles at the level of localized sources. Therefore, the general solutions must be adapted to specific scenarios of source generation phenomenon.

3.1 Scenarios

First, we assume that the particle generation process takes place in the frame of a thin geometry, that is, the acceleration efficiency is high enough to ignore the collisional deceleration efficiency of the medium. Even when this is not true everywhere, and a thick geometry prevails sometimes in localized solar particle sources, this assumption avoids complications. What we are trying to do is study the effects of energy diffusion relative to the so called average systematic energy change. Also for the sake of simplicity we assume that the parameter τ is time and energy independent.

In the case of solar flare particle generation, most events are commonly observed to be associated either with one or two acceleration stages. Therefore we consider the following two scenarios. First, particles are impulsively accelerated from the thermal background up to high energies, with injection at time $t = 0$. We assume that this acceleration stage is ruled by the above-mentioned stochastic processes, though this may not be necessarily true: in the literature it is usually associated with direct DC electric field acceleration. The second scenario occurs when particles of the thermal background are preliminary accelerated to moderately high energies and an important fraction of this population is continuously injected into an adjacent volume, where the stochastic process reaccelerates them up to high energies. It is usually assumed that the primary acceleration step in this scenario is deterministic and is associated with magnetic reconnection in a magnetic neutral current sheet. We also suppose that the continuous injection is independent of time $q(E,t) = q(E)\Theta(t)$, where $\Theta(t)$ is a step function such that $\Theta(t) = 1$ for $t > 0$, and $\Theta(t) = 0$ for $t \leq 0$. According to Pérez-Peraza *et al* (1978), the energy spectrum from neutral current sheet acceleration is given by

$$q(E) = 9.1 \times 10^{-2}\, nV_d\, (E/E_c)^{-1/4} \exp[-1.12(E/E_c)^{3/4}]$$

(part/eV cm² s sr)

where E denotes particle kinetic energy, $E_c = (eLBV_d\, m^{1/2}/2c)^{2/3}$, n = number particle density, L = length of the neutral current sheet, m, e are the particle mass and the electronic charge respectively and $V_d = 0.057\, V_a$ is the diffusion velocity between the magnetic field lines and matter according to the model by Priest (1973). By changing units we have

$$q(E) = 1.27 \times 10^4 N_{Td} B (m/n)^{1/2} \left(\frac{E_c^{1/4}}{E^{3/4}}\right) \exp\left[-1.12\left(\frac{E}{E_c}\right)^{3/4}\right]\left(\frac{Part}{eV}\right) \qquad (15)$$

where N_{Td} is the total number of particles in the diffusion volume. This was evaluated by setting q(E) = 1 part/eV in E = Eo (the threshold energy for stochastic acceleration).

For the second scenario, it is assumed that only particles injected from the preliminary acceleration step which fulfill the threshold conditions of the stochastic process participate in the secondary acceleration stage. We exclude the particles of the local thermal population that might fulfill the resonant condition and threshold energy for stochastic acceleration. This is an idealization of the real physical situation in order to simplify the mathematical analysis. For both scenarios we consider two different assumptions: a kind of "smoothed situation" for the generation phenomenon, where, to some extent, a steady-state situation is established, and a more dynamic situation, where the generation process is strongly time-dependent (non-stationary). For the first scenario, and for the second stage of the second scenario, both the classical Fermi mechanism and turbulent acceleration by magnetosonic waves are considered.

3.2 Energy spectra without energy diffusion

This assumption is obtained by setting $\langle dE^2/dt \rangle = 0$ in equation (6),

$$\frac{\partial N(E,t)}{\partial t} + \frac{\partial}{\partial E}\left[\left\langle \frac{dE}{dt} \right\rangle N(E,t)\right] + \frac{N(E,t)}{\tau(E,t)} = Q(E,t) \qquad (16)$$

The solution of this equation determines the contribution of the systematic energy gain in the generation of energetic particles.

3.2.1 Stationary Solution

The solution of eq. (16) when $Q(E,t) = q(E)$ and $\tau(E,t) = \tau$ is as follows

$$N(E) = N_L(E_o) \exp(-t^*/\tau)/A(E)$$

$$+A^{-1}(E)\int_{E_o}^{E} q(E')exp[(t'^*-t^*)/\tau]dE' \quad (17)$$

where E_o is the threshold energy imposed by the intrinsic interaction properties of the acceleration mechanism, N_L is the initial population in the acceleration volume evaluated at the specific energy value E_o, $A(E) = <dE/dt>$ is the systematic acceleration rate (without energy losses) discussed in Section 2, $A(E_o)$ is the same rate evaluated for E_o and $t^* = \int_{E_o}^{E} dE'/A(E')$, is the acceleration time from the threshold injection value up to energy E. The first term in eq. (17) is the contribution of particles of local matter to the energy spectrum (1st scenario) and the second term quantifies the contribution of particles from a preliminary acceleration stage to the stochastic acceleration process (second scenario).

3.2.1.1a. First Scenario with Fermi Acceleration

The solution in this case is obtained by evaluating t in the first term of (17) with the specific rate $A(E) = \alpha_f\beta W$ as given in eq. (9)

$$N(E)=N_{LT}(E_o)\left(\frac{\beta_o W_o}{\beta W}\right)\left[\frac{W+\left(W^2-m^2c^4\right)^{1/2}}{W_o+\left(W_o^2-m^2c^4\right)^{1/2}}\right]^{-\frac{1}{\alpha_f\tau}} \quad (18)$$

where $N_{LT}(E_o) = 2\pi N_o E_o^{1/2}exp(-E_o/KT)/(\pi KT)^{3/2}$ is the number of thermal particles of specific energy E_o, N_o is the net number of thermal particles with $E \geq E_o$, able to participate in the acceleration process and KT is the Boltzmann thermal energy.

3.2.1.1b. Second Scenario with Fermi Acceleration

From the second term of eq. (17) it follows, that

$$N(E)=\left(\alpha_f\beta W\right)^{-1}$$

$$\int_{E_o}^{E} q(E')\left[\frac{W+\left(W^2-m^2c^4\right)^{1/2}}{W'+\left(W'^2-m^2c^4\right)^{1/2}}\right]^{-\frac{1}{\alpha_f\tau}} dE' \quad (19)$$

where the threshold energy E_o refers now to particles of the suprathermal injection spectrum q(E), since the local particles were excluded.

3.2.1.2a. First Scenario with Turbulent Acceleration

In this case t^* is evaluated with eq. (12), $A(E) = \alpha_m\beta^2 W$. From the first term of eq.(17) we find

$$N(E)=N_{LT}(E_o)\left(\frac{\beta_o^2 W}{\beta^2 W}\right)\left[\frac{W^2-m^2c^4}{W_o^2-m^2c^4}\right]^{-\frac{1}{2\alpha_m\tau}} \quad (20)$$

where $\beta = (W^2-m^2c^4)^{1/2}/W$, and $\beta_o = \beta\,(E_o)$.

3.2.1.2b. Second Scenario with Turbulent Acceleration

From the second term of eq. (17) we have,

$$N(E)=(\alpha_m\beta^2 W)^{-1}$$

$$\int_{E_o}^{E} q(E')\left[\frac{W^2-m^2c^4}{W_o^2-m^2c^4}\right]^{-\frac{1}{2\alpha_m\tau}} \cdot dE' \quad (21)$$

Here again the acceleration threshold value E_o is in the domain of q(E).

3.2.2. TIME DEPENDENT SOLUTION

The solution of equation (16) in this case may be derived by the Laplace transform method (e.g. Melrose, 1976), which yields the following result

$$N(E,t)=N_{LT}(E_i)A(E_i)exp(-t/\tau)/A(E)$$

$$+A^{-1}(E)\int_{E_o}^{E} q(E')exp[(t^{*\prime}-t^*)/\tau]dE' \quad (22)$$

where the initial particle energy E_i (t) is derived from the acceleration time $t=\int_{E_i}^{E} dE'/A(E')$, with $E_o<E_i<E$, such that at $t = 0$, $E_i(t) = E_o$. Here, $N_{LT}(E_i,0)$ is a pulse of particles of the thermal distribution evaluated from E_i up to infinity at $t = 0$. Here again, the first and second terms in eq. (22) correspond to the first and second scenarios.

3.2.2.1a. First Scenario with Fermi Acceleration

Setting the acceleration rate (11) as $A(E) = \alpha_f\,(W^2-m^2c^4)^{1/2}$, the acceleration time is $t = \{[W+(W^2-m^2c^4)^{1/2}]/[W_i+(W_i^2-m^2c^4)^{1/2}]\}^{1/\alpha_f}$, with $W_i = mc^2(1-\beta_i^2)^{-1/2}$ and $\beta_i=$

$[1+(\beta^{-2}-1)\exp(2\alpha t)]^{-1/2}$, so that the spectrum may be written in the short form

$$N(E,t)=\frac{2\pi N_o}{(\pi KT)^{3/2}}\left[\frac{W_i^2-m^2c^4}{W^2-m^2c^4}\right]^{1/2} E_i^{1/2}\exp\left[-\left(\frac{t}{\tau}+\frac{E_i}{KT}\right)\right] \quad (23)$$

where W_i is defined within the local thermal distribution with $W_i = W_o$ at t= 0.

3.2.2.1b. Second Scenario with Fermi Acceleration

By substitution of t* and t*' in the second term of eq. (22), we obtain,

$$N(E,t)=\left(\alpha_f\beta W\right)^{-1}\int_{Ei}^{E} q(E')\left[\frac{W'+(W'^2-m^2c^4)^{1/2}}{W+(W^2-m^2c^4)^{1/2}}\right]^{-\frac{1}{\alpha_f\tau}} dE' \quad (24)$$

where E_i is in the domain of the injection spectrum q(E) with $E_i = E_o$ at t = 0.

3.2.2.2a. First Scenario with Turbulent Acceleration

In this case we evaluate the acceleration time t(E) in the first term of eq. (22) by setting $W_i = [m^2c^4 + (W^2-m^2c^4)\exp(-2\alpha_m)]^{1/2}$, thus, the spectrum is

$$N(E,t)=\frac{2\pi N_o}{(\pi KT)^{3/2}}E_i^{1/2}\left(\frac{\beta_i^2W_i}{\beta^2W}\right)\exp\left[-\left(\frac{t}{\tau}+\frac{E_i}{KT}\right)\right] \quad (25)$$

where $\beta_i = \beta(W_i)$.

3.2.2.2b. Second Scenario with Turbulent Acceleration

Following the same procedure as in eq. (24) we obtain

$$N(E,t)=\left(\alpha_m\beta^2W\right)^{-1}\int_{Ei}^{E} q(E_i)\left[\frac{W^2-m^2c^4}{W'^2-m^2c^4}\right]^{-\frac{1}{2\alpha_m\tau}} dE' \quad (26)$$

3.3 Energy spectra with energy diffusion

In this case we are dealing with the solution of equation (6) including the term of energy spread, i.e., the second term of the right hand of (6).

3.3.1. Stationary solution

By setting $\partial N(E,t)/\partial t = 0$ in equation (6), we obtain the reduced form which is solved by the WKBJ method (appendix A). The result is

$$N(E)=N_{LT}(E_o)\left[\frac{D(E)}{D(E_o)}\right]^{1/4}\exp\left\{\frac{1}{2}\int_{Eo}^{E}\left(\frac{A}{D}+\frac{1}{D}\frac{dA}{dE}\right)dE' -a^{1/2}\int_{Eo}^{E}\frac{dE'}{D^{1/2}(E')}\right\}$$

$$+\frac{D^{1/4}(E)}{2a^{1/2}}\int_{Eo}^{E}\frac{q(E')}{D^{3/4}(E')}\exp\left\{\frac{1}{2}\int_{E'}^{E}\left(\frac{A}{D}+\frac{1}{D}\frac{dA}{dE''}\right)dE'' -a^{1/2}\int_{E'}^{E}\frac{dE''}{D^{1/2}(E'')}\right\}dE'\left(\frac{Part}{Energy}\right) \quad (27)$$

where $E_o \le E' \le E$, $a = (1/\tau) + \overline{F} = (1/\tau) + \overline{(dA(E)/dE - (d^2/dE^2)\,D(E)}$.

The approximation made by introducing $\overline{F}$ in a leads to an analytical solution of the corresponding integrals: for the range of the relevant parameters $\alpha\tau$ and E_o values in solar particle production, the deviation of the approximation from the numerical solution for protons (e.g. Miller *et al*, 1990) is about 1% and 5% for electrons. The intrinsic relative error of the WKBJ method (relative to a solution with terms of third order or higher) is less than 0.1% for protons and 1% for electrons (Figure 11).

3.3.1.1a. First Scenario with Fermi Acceleration

The energy spectrum in this case is obtained by substituting A(E) and D(E) in the first term of eq. (27), so, that according to appendix A (equation A.17)

$$N(E)=N_{LT}(E_o)\left(\frac{W}{W_o}\right)^{3/4}\left(\frac{\beta_oW_o}{\beta W}\right)^{1/8}\exp\left[-2(a/\alpha_f)^{1/2}J(E_o,E)\right] \quad (28)$$

where N_{LT} is the number of particles defined by the evaluation of the thermal spectrum at the specific energy E_o, and $J(E_o,E) = \tan^{-1}\beta^{1/2} - \tan^{-1}\beta_o^{1/2} + (1/2)\ln[(1+\beta^{1/2})(1-\beta_o^{1/2})/(1-\beta^{1/2})(1+\beta_o^{1/2})]$

3.3.1.1b. Second Scenario with Fermi Acceleration

From the second term of eq. (27) the following spectrum is obtained:

$$N(E)=\left(\frac{3}{2a\alpha_f}\right)^{1/2}\left(\frac{W^{1/2}}{\beta^{1/4}}\right)$$

$$\int_{E_o}^{E}\frac{q(E')}{\beta'^{5/4}W'^{3/2}}exp\left[-\left(\frac{3a}{\alpha_f}\right)^{1/2}J(E',E)\right]dE \tag{29}$$

where J(E',E) is similar to J(E_o ,E) as defined in (28), but with $\beta' = \beta_o$.

3.3.1.2a. First Scenario with Turbulent Acceleration

Introducing in eq. (27) the corresponding A(E) and D(E) we obtain, according to equation (A.18) of appendix A,

$$N(E)=N_{LT}(E_o)\left(\frac{\beta_o}{\beta}\right)^{11/4}\left(\frac{W_o}{W}\right)^{5/4}\left[\frac{\beta W}{\beta_o W_o}\right]^{-\left(\frac{a}{2\alpha_m}\right)^{1/2}} \tag{30}$$

where N_{LT} has the same meaning as in eq. (28).

3.3.1.2b Second Scenario with Turbulent Acceleration

From the second term of eq. (27) we find in this case,

$$N(E)=\left(\frac{3}{2a\alpha_m}\right)^{1/2}\left(\beta^{11/4}W^{5/4}\right)^{-1}$$

$$\int_{E_o}^{E}q(E')\beta'^{3/4}\,W'^{1/4}\left(\frac{\beta W}{\beta' W'}\right)^{-\left(\frac{a}{2\alpha_m}\right)^{1/2}}dE' \tag{31}$$

3.3.2. Time-dependent solution

By applying Laplace transforms in equation (6), and following the procedure of Appendix B, the non-stationary solution is

$$N(E,t)=\frac{D^{1/4}(E)}{2(\pi t)^{1/2}}exp\left[-at-\frac{1}{2}\int_{E_o}^{E}P_1dE'\right]$$

$$\int_{E_o}^{E}\frac{N(E',0)}{D^{3/4}(E')}exp\left\{\frac{3}{2}\int_{E_o}^{E'}P_1dE''-\frac{1}{4t}\left[\int_{E''}^{E}\frac{dE''}{D^{1/2}(E'')}\right]^2\right\}dE'$$

$$+\frac{D^{1/4}(E)}{(4\pi)^{1/2}}exp\left[-\frac{1}{2}\int_{E_o}^{E}P_1dE'\right]\int_o^t\frac{dt'}{t'^{1/2}}\int_{E_o}^{E}\frac{q(E')}{D^{3/4}(E')}$$

$$exp\left\{-at'+\frac{3}{2}\int_{E_o}^{E}P_1dE'-\frac{1}{4t'}\left[\int_{E'}^{E}\frac{dE''}{D^{1/2}(E'')}\right]^2\right\}dE'\left(\frac{Part}{Energy}\right) \tag{32}$$

where $P_1 = -A(E)/D(E) + (2/D)(dD/dE)$. The first term in (32) represents the contribution of local particles to the acceleration spectrum (first scenario), and the second term is the contribution of the injected particles from a preliminary acceleration stage (second scenario). As in the previous cases E_o in the first term is defined within the thermal population, whereas in the second term it is defined within the injected energetic spectrum q(E). The analytical solution of the integrals in time has been given in Pérez-Peraza and Gallegos-Cruz (1994).

3.3.2.1a First Scenario with Fermi Acceleration

Introducing in the first term of eq. (32) the corresponding expressions for A(E) and D(E), and using the same integration as in eq. (A.17) of Appendix A, the resulting spectrum is

$$N(E,t)=\left(\frac{3}{4\pi\alpha_f t}\right)^{1/2}\frac{W^{1/2}}{\beta_o^2\beta^{1/4}}exp(-at)$$

$$\int_{E_o}^{E}\frac{N(E',0)\beta'^{3/4}}{W'^{3/2}}exp\left[-\left(\frac{3}{4\alpha_f t}\right)J^2(E',E)\right]dE' \tag{33}$$

where J(E_o ,E) is as defined in eq. (28) and N(E',0) is the initial particle thermal distribution at t = 0, given as $N(E',0) = [2\pi N_o/(\pi KT)^{3/2}]\exp(-E'/kT)$, KT = Boltzmann thermal energy and $N_o = \int_{E_o}^{\infty} N(E',0)dE'$ (as in eq. 18) is the total number of particles of the Maxwellian distribution with $E \geq E_o$ participating in the acceleration process.

3.3.2.1b Second Scenario with Fermi Acceleration

From the second term of eq. (32) we obtain

$$N(E,t)=\left(\frac{3}{4\pi\alpha_f}\right)^{1/2}\frac{W^{1/2}}{\beta_o^2\beta^{1/4}}\int_o^t\frac{dt'}{t'^{1/2}}$$

$$\int_{E_o}^{E}\frac{q(E')\beta'^{3/4}}{W^{3/2}}exp\left\{-at'-\left(\frac{3}{4\alpha_f t}\right)J^2(E',E)\right\}dE' \tag{34}$$

3.3.2.2a. First Scenario with Turbulent Acceleration

By introducing the rates A(E) and D(E) in the first term of eq. (32), and using Appendix B, we obtain:

$$N(E,t)=\frac{(8\pi\alpha_{m}t)^{-1/2}exp(-at)}{\beta_o^{15/2}\beta^{11/4}W_o^{7/2}W^{5/4}}$$

$$\int_{E_o}^{E}N(E',0)\beta'^{33/4}W'^{15/4}exp\left\{-\frac{1}{2\alpha_{m}t}\left[ln\left(\frac{\beta W}{\beta' W'}\right)\right]^2\right\}dE' \tag{35}$$

3.3.2.2b. Second Scenario with Turbulent Acceleration

In this case from the second term of eq. (32) we obtain

$$N(E.t) = \frac{(8\pi\alpha_m)^{-1/2}}{\beta_o^{15/2}\beta^{11/4}W_o^{7/2}W^{5/4}} \int_o^t \frac{dt'}{t'^{1/2}} \int_{Eo}^{E} q(E')\beta'^{33/4}W'^{1\ 5/4}$$

$$exp\left\{-at' - \frac{1}{2\alpha_{ml}}\left[ln\left(\frac{\beta W}{\beta' W'}\right)\right]^2\right\} dE' \quad (36)$$

4. RESULTS

For a quantitative evaluation of the relevance of energy diffusion during particle acceleration in the formation of the energy spectrum, we define the ratio (R) between the complete energy spectrum when the systematic rate and the rate of diffusion in energy are included, and the energy spectrum when only the systematic rate is included. Thus, if R = 1, the effects of the spread around the average energy gain are not relevant in the formation of the spectrum. If the contribution of these effects is ignored the particle flux· is underestimated by ~ 50%. If **R** >> 1, there is a strong particle overproduction per energy band due to dispersive effects relative to the average acceleration rate. If **R** << 1, then the spread around the average rate leads to a relative depression per energy band of particle flux. If ignored, the flux in particle spectra is strongly overestimated. The ratio **R** as defined makes our results independent of the initial number of particles that participate in the acceleration process (i.e., the number N_o in the first scenario through the Mawellian distribution, and the factor nV_d in the injection spectrum q(E) in the second scenario). No differences in absolute flux are evaluated with **R**, but only relative differences in flux in different energy bands.

For turbulent magnetosonic wave acceleration the initial threshold injection condition may be taken roughly as $v > V_a$, $E_o = E(V_a)$, therefore, for purposes of comparison in the first scenario, we also take for the Fermi mechanism the value $E_o = (1/2)m\,V_a^2$ and we calculate V_a with B = 50 gauss and n = 10^{10} cm^{-3}. Also in the first scenario, for the nonstationary case where the parameter temperature appears explicitly, we arbitrarily use T = (2.5- 5)x10^8 K as in Tables 1 and 2. For the second scenario we assume that the preliminary acceleration stage takes place in a magnetic neutral current sheet, where B = 500 gauss, L= 10^9 cm, and n = 10^{12} cm^{-3}. For E_o which appears in the spectra of the second scenario, we arbitrarily use (1-50)KeV for electrons and (10-50) Kev for protons, as in Tables 1 and 2. As for the acceleration time t which appears in the derived spectra for the nonstationary case, we use typical times for solar particle acceleration as found in the literature (0.1s-60s). Whenever there is a value in the column for the time t in Tables 1 and 2, we are dealing with the nonstationary case; otherwise the columns and curves refer to the stationary case.

Tables 1 and 2 summarize the parameters used in the calculations. The analysis applies to the following eight cases:

(1) The stationary case of the first scenario, with **R** = eq.(28)/eq.(18) for the Fermi process (Figures 1a, 3a and 3b).

(2) The stationary case of the second scenario for the Fermi process, **R** = eq. (29)/eq. (19), (Figures 1c, 5a, 5b and 5c).

(3) The stationary case of the first scenario, for turbulent acceleration by magnetosonic waves, **R** = eq. (30)/eq. (20) (Figures 2a, 7a and 7b).

(4) The stationary case of the second scenario for turbulent acceleration by magnetosonic waves **R** = eq. (31)/eq. (21) (Figures 2c, 9a, 9b and 9c).

(5) The non-stationary case of the first scenario for Fermi acceleration, **R** = eq. (33)/eq. (23) (Figures 1b, 4a, 4b, 4c and 4d).

(6) The non-stationary case of the second scenario for the Fermi process, **R** = eq. (34)/eq. (24) (Figures 6a, 6b and 6c).

(7) The non-stationary case of the first scenario for turbulent acceleration by magnetosonic waves, **R** = eq. (35)/eq. (25) (Figures 2b, 8a, 8b, 8c and 8d).

(8) The non-stationary case of the second scenario for turbulent acceleration by magnetosonic waves, **R** = eq. (36)/eq. (26) (Figures 10a, 10b and 10c). The results are given in two categories: global findings and fine structure findings.

4.1. Global findings

(1) R is energy-dependent and may vary between 1 and several orders of magnitude (Figures 1-10).

(2) For a given scenario and acceleration process, the effects of the energy spread are more important in the nonstationary case than in the stationary case. This can be seen in curves 1a/1b, or 2a/2b of Figure 1. However, for some sets of parameters in the stationary case, there are values of **R** ~ 1, then the incomplete solution can be approximated within an order of magnitude to the complete solution.

(3) For either case (stationary or nonstationary), the diffusion in energy is more important in the first scenario, when low energy particles are involved, than in the second scenario. This is more noticeable for the high-energy portion of the spectra, as can be seen in Figures 1a/1c, or 2b/2d.

Table 1

Parameters in the Fermi Acceleration Process

FIG.	CURVE	SCENARIO	ELECTRONS $\alpha_f(s^{-1})$	$\tau(s)$	$T(^oK)$	E_o (eV)	t(s)	PROTONS $\alpha(s^{-1})$	$\tau(s)$	$T(^oK)$	E_o (eV)	t(s)
1	a	1st	-	-	-	-	-	0.19	0.5	-	-	-
	b	1st	-	-	-	-	-	0.19	0.5	5×10^8	-	0.1
	c	2nd	-	-	-	-	-	0.19	0.5	-	5×10^4	-
	d	2nd	-	-	-	-	-	0.95	0.1	-	-	0.1
3	a	1st	0.164	0.5	-	-	-	0.164	0.5	-	-	-
	b	1st	0.198	0.5	-	-	-	0.124	0.5	-	-	-
4	a	1st	0.194	0.5	5×10^8	-	0.1	0.194	0.5	5×10^8	-	0.1
	b	1st	0.014	0.5	5×10^8	-	0.1	0.096	0.5	5×10^8	-	0.1
	c	1st	0.194	0.5	2.5×10^8	-	0.1	0.194	0.5	2.5×10^8	-	0.1
	d	1st	0.194	0.5	5×10^8	-	12.	0.194	0.5	5×10^8	-	0.25
5	a	2nd	0.186	0.5	-	10^4	-	0.194	0.5	-	5×10^4	-
	b	2nd	0.052	0.5	-	10^4	-	0.096	0.5	-	5×10^4	-
	c	2nd	0.186	0.5	-	10^3	-	0.194	0.5	-	10^4	-
6	a	2nd	0.194	0.5	-	-	0.5	0.75	0.1	-	-	0.1
	b	2nd	0.104	0.5	-	-	0.5	0.10	0.1	-	-	0.1
	c	2nd	0.194	0.5	-	-	0.25	0.75	0.1	-	-	17.

(4) The importance of diffusion in energy for spectrum formation is markedly similar for both acceleration processes, though slight differences occur at the high energy end of the spectrum.

(5) The energy diffusion effects are similar for electrons and for protons, except for a shift in energy due to the difference in mass.

4.2. Fine structure findings

(1) The effects of energy diffusion tend to be minimized in the formation of the energy spectrum ($\mathbf{R} \rightarrow 1$) as the acceleration efficiency, α, increases (within the limits discussed in Appendix C). Thus, if $\mathbf{R} < 1$ in a given energy range, $\mathbf{R}$ increases with α, but if $\mathbf{R} > 1$ then $\mathbf{R}$ decreases with increasing α.

(2) The effects of diffusion in energy in the time-dependent case tend to be minimized ($\mathbf{R} \rightarrow 1$) as the acceleration time elapses, with some tendency toward the steady-state situation. This can be seen in curves 4a/4d, 6a/6c, 8a/8d and 10a/10c.

(3) In the first scenario, there is a minimization of energy diffusion effects ($\mathbf{R} \rightarrow 1$) at relatively high energies, as the background temperature T increases. This can be seen in curves 4a/4c and 8a/8c.

(4) In the second scenario, the effects of diffusion at low energies are minimized ($\mathbf{R} \rightarrow 1$) as the initial threshold injection value E_o decreases. This can be seen in curves 5a/5c and 9a/9c.

5. DISCUSSION

In spatial diffusion, particle encounters with em inhomogeneities produce changes in the direction they would have followed along the background magnetic field lines if no inhomogeneities had been present. Similarly, diffusion in energy space during particle acceleration may be seen as a change of direction in the energy gain process, relative to the deterministic energy change process for a systematic energy gain rate, in the absence of energy spread. A change of direction in energy space means a slowdown or a speeding-up in effective energy gain rate relative to an average acceleration rate (here designated as the systematic rate).

Table 2

Parameters in Magnetosonic Wave Acceleration

			PARAMETERS									
FIG.	CURVE	SCENARIO	ELECTRONS					PROTONS				
			$\alpha_m(S^{-1})$	$\tau(S)$	$T(^oK)$	E_o (eV)	t(S)	$\alpha(S^{-1})$	τ (S)	$T(^oK)$	E_o (eV)	t(S)
2	a	1st	-	-	-	-	-	0.188	0.5	-	-	-
	b	1st	-	-	-	-	-	0.188	0.5	5×10^8	-	0.15
	c	2nd	-	-	-	-	-	0.235	0.4	-	3×10^4	-
	d	2nd	-	-	-	-	-	0.188	0.5	-	-	0.15
7	a	1st	0.082	0.5	-	-	-	0.082	0.5	-	-	-
	b	1st	0.132	0.5	-	-	-	0.048	0.5	-	-	-
8	a	1st	0.116	0.5	5×10^8	-	0.1	0.194	0.5	5×10^8	-	0.1
	b	1st	0.078	0.5	5×10^8	-	0.1	0.156	0.5	5×10^8	-	0.1
	c	1st	0.116	0.5	2.5×10^8	-	0.1	0.194	0.5	4×10^8	-	0.1
	d	1st	0.116	0.5	5×10^8	-	0.07	0.194	0.5	5×10^8	-	0.15
9	a	2nd	0.194	0.5	-	10^4	-	0.24	0.4	-	3×10^4	-
	b	2nd	0.074	0.5	-	10^4	-	0.037	0.4	-	3×10^4	-
	c	2nd	0.194	0.5	-	5×10^4	-	0.24	0.4	-	10^4	-
10	a	2nd	0.198	0.5	-	-	0.25	0.188	0.5	-	-	0.25
	b	2nd	0.116	0.5	-	-	0.25	0.058	0.5	-	-	0.25
	c	2nd	0.198	0.5	-	-	10.	0.188	0.5	-	-	0.15

Diffusion in energy implies the escape of particles from some energy bands (**R** < 1) and their accumulation in other energy bands (**R** > 1) relative to the distribution from an unidirectional systematic acceleration rate. Thus, energy diffusion may lead to particle loss if some particles diffuse to low energy bands below the acceleration threshold E_o. On the other hand, it may also lead to energetic particle generation, because particles of subsidiary resonant bands may diffuse into the central resonant band associated to the average phase velocity of each resonant wave, so even particles from energy bands below E_o may diffuse into bands above E_o when the variance of the frequency of resonant wave-particle interactions is very high.

The way we have defined **R**, the ratio of particle flux, only allows us to analyze the depletion or overabundance of particles per energy band; the information about the net number of particles is lost.

Suppose that the combined effect of the average systematic rate and the rate of diffusion in energy may be seen as a kind of effective energy gain rate $(dE/dt)_{eff} \cong \alpha_{eff}\beta^n W^m$ (n real, m > 0). This effective rate may be similar, slower or faster than the average systematic rate in a given energy range, so that it may generate similar (**R**~1), lower (**R** < 1) or higher (**R** > 1) number of particles in an energy range.

Many factors contribute to whether (**R** >1, ~1, < 1), as well as the magnitude of **R**: the initial population (scenario), depending on the sign of $\partial f/\partial v$ as discussed in section 2.2, and on the value E_o, the temporal (steady-state or time-dependent) behavior of the particle generation phenomenon, and the involved acceleration process. The main factor is the acceleration efficiency of the process (more exactly, the value of the parameter product $\alpha\tau$ appearing in the spectra). The average efficiency of the systematic rate in the Fermi mechanism is determined by the ratio of accelerative interactions ("head-on" collisions) to decelerating interactions ("catch-up" collisions). If the relative proportion of head-on collisions increases or decreases with respect to the average then, α_{eff} will increase or decrease respectively, and so will **R**. In the first approximation, high values of acceleration efficiency are due to high frequencies of accelerating head-on collisions, and viceversa. Also, according to equations (9) and (14), the higher or lower values of α_f and α_m in Tables 1 and 2 depend on the related

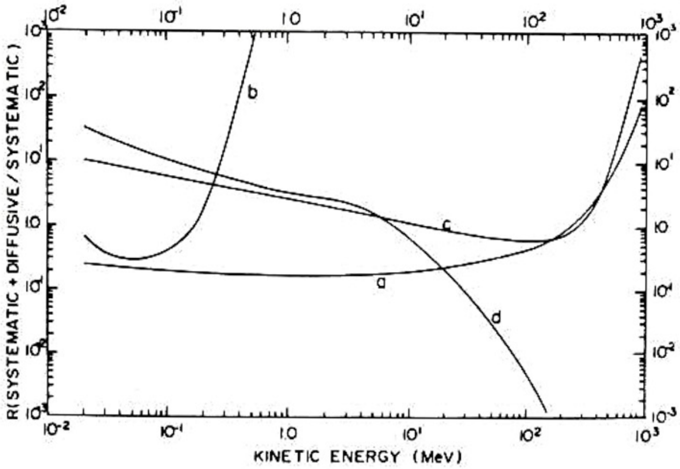

Fig. 1. The ratio R for the Fermi process: (a) stationary first scenario, (b) non-stationary first scenario, (c) stationary second scenario, (d) non-stationary second scenario. All four cases are evaluated with the same parameters (Table 1).

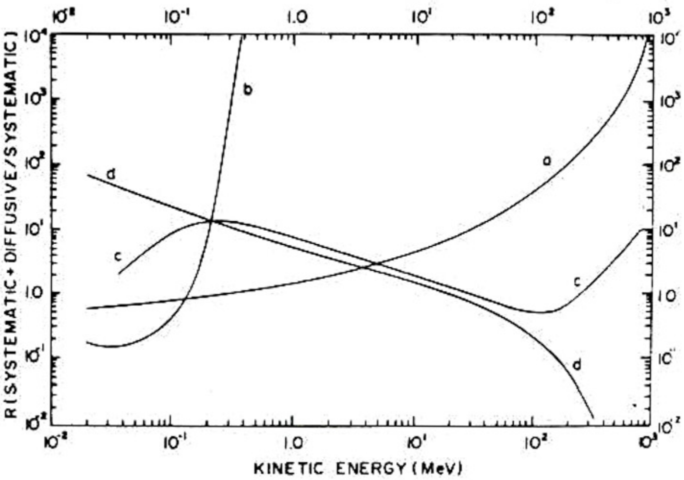

Fig. 2. The ratio R for Magnetosonic Wave Acceleration: (a) stationary first scenario, (b) non-stationary first scenario, (c) stationary second scenario, (d) non-stationary second scenario. All four cases are evaluated with the same parameters (Table 2).

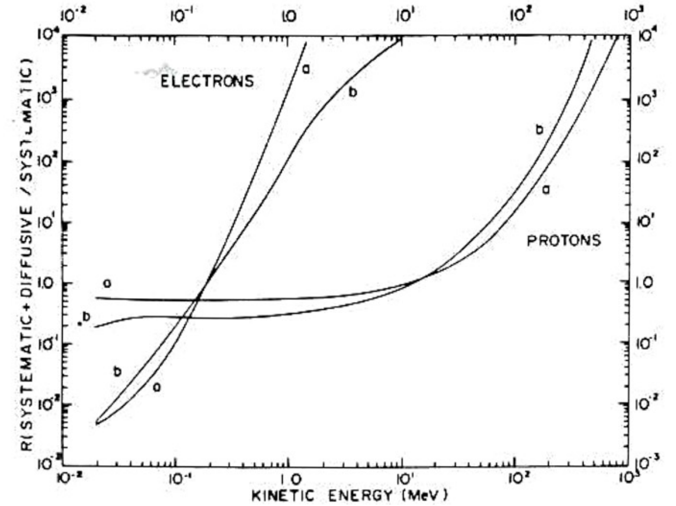

Fig. 3. The ratio R for the Fermi process in the stationary case of the first scenario. The parameters used for each curve are given in Table 1.

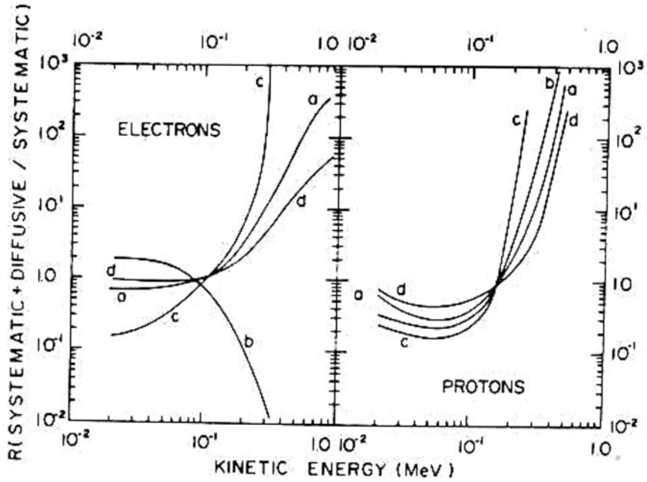

Fig. 4. The ratio R for the Fermi process in the non-stationary case of the first scenario. The parameters are given in Table 1.

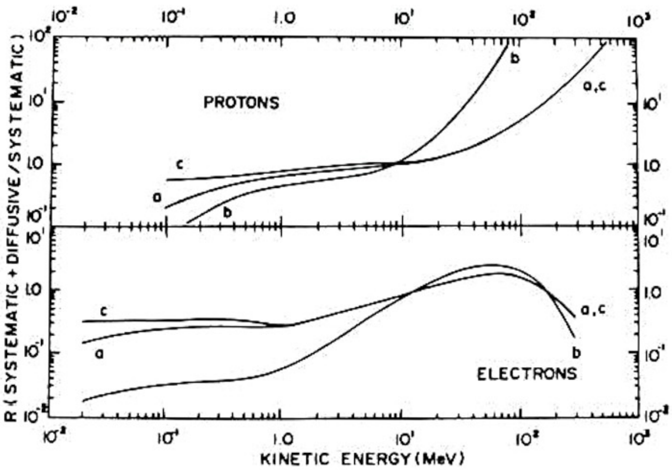

Fig. 5. The ratio R for the Fermi process in the stationary case of the second scenario. The parameters are given in Table 1.

values of the hydromagnetic and wave phase velocities, on the size scale of the inhomogeneities and magnetosonic waves, and on the magnetic field strength.

Recall from statistics that the higher the number of events the lower is the variance and the dispersion around the average value of a parameter. Here the events are interactions between particles and accelerating agents. The number of interactions is proportional to the number of "resonant" particles per energy band. Hence the importance of diffusion effects in a given energy band is related to the number of "resonant" particles, which in turn is related to the acceleration efficiency. In fact, as stated in 4.2(1), when α increases the effects of the energy spread on the spectra tend to become less important. This can be seen, for instance, in the Fermi process, where an increase in α_f is associated with an increase in turbulence which results in an increase of the local hydromagnetic velocity U and a decrease of the mean distance l between the accelerating agents ($\alpha_f \sim U^2/l$). This entails an increase in the frequency of accelerating collisions and a reduction of the statistical variance. Consequently the dispersion $\langle dE^2/dt \rangle$ around the

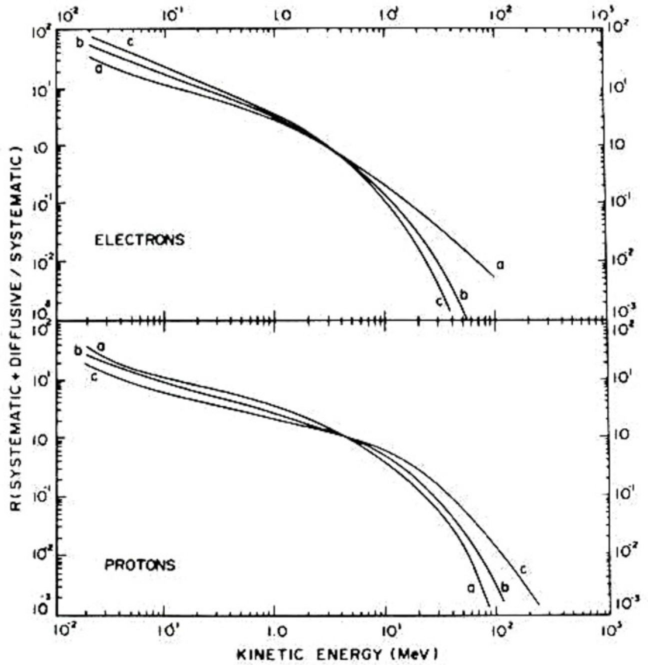

Fig. 6. The ratio R for the Fermi process in the non-stationary case of the second scenario. The parameters are given in Table 1.

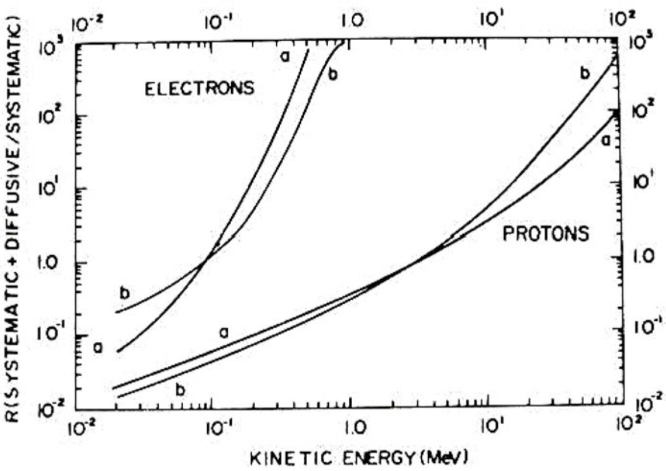

Fig. 7. The ratio R for Magnetosonic Wave Acceleration in the stationary case of the first scenario. The parameters are given in Table 2.

average rate $\langle dE/dt\rangle_{sys}$ is small. Therefore the absolute magnitude of **R** (separation from the unit) is highly dependent on the acceleration efficiency. The same can be said for turbulent acceleration by magnetosonic waves; in fact, according to equation (14) ($\alpha_m \sim V^2_\phi/B^2$) a change in the value of α_m is directly related to a change in the phase velocity V_ϕ . An increase in V_ϕ means that, among the group of particles that satisfy the resonant condition (section 2.2), the relative proportion of particles where $v \leq V_\phi$ increases over those where $v > V_\phi$. This entails a higher ratio of accelerated particles (head-on collisions) to decelerated particles (catch-up collisions). In other words, the frequency of accelerating interactions increases with the number of resonant particles. Hence the variance of the energy gain rate decreases, that is, the dispersion around the average value of the rate decreases. In conclusion, the effect of diffusion in energy gradually decreases as α_m increases. An equivalent form of visualizing this argument is as follows. As we saw in section 2.2, the resonant particles that satisfy the condition $v_{\parallel} = (k/k_{\parallel})V_a$, with $k > k_{\parallel}$ ($v_{\parallel} > V_a$) are susceptible to acceleration. Since $V_a \propto B$, an increase in α_m (which entails a decrease in B) is associated with a decrease in V_a, implying an increase in the number of particles that satisfy the condition $v > V_a$, and thus an increase in the frequency of wave-particle interactions and a decrease in diffusion effects. Therefore, it appears that the importance of diffusion effects must be related to the acceleration efficiency, which produces a larger or smaller accumulation of resonant interacting particles per energy band. Since all these effects are initially determined in the low energy region of the spectra, the importance of diffusion (absolute value of **R** in Figures 1-10) in the high energy region of the spectra is a compensating effect for what was lost or gained in the low energy bands.

The difference between the results obtained in the first and second scenarios [see 4.1(3)] may be understood as follows. In the first scenario the initial population is thermal and acceleration starts from very low energies. In contrast, in the second scenario the injection threshold lies within the energetic population of the spectrum q(E). The low-energy population in the first scenario is distributed over a relatively wide range of energy bands, with few particles in

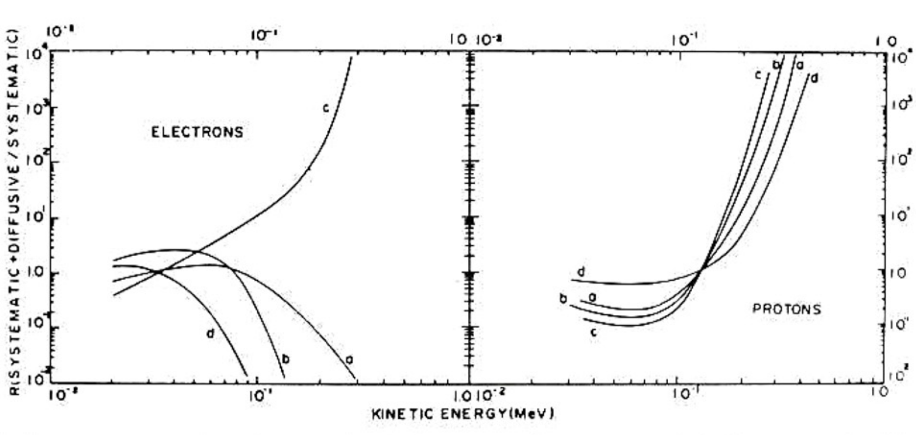

Fig. 8. The ratio R for Magnetosonic Wave Acceleration in the non-stationary case of the first scenario. The parameters are given in Table 2.

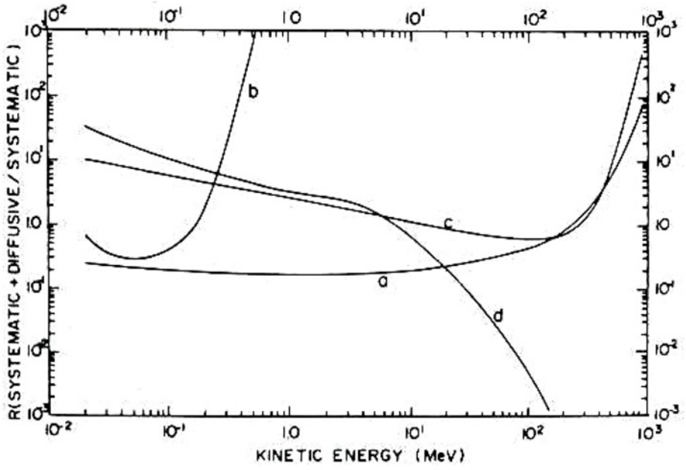

Fig. 1. The ratio R for the Fermi process: (a) stationary first scenario, (b) non-stationary first scenario, (c) stationary second scenario, (d) non-stationary second scenario. All four cases are evaluated with the same parameters (Table 1).

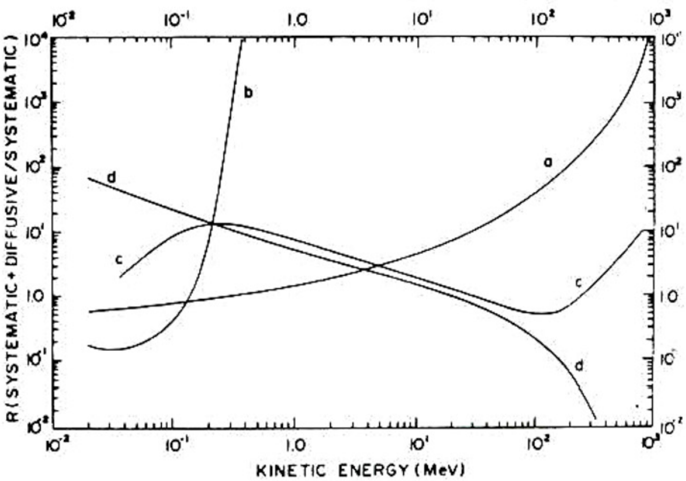

Fig. 2. The ratio R for Magnetosonic Wave Acceleration: (a) stationary first scenario, (b) non-stationary first scenario, (c) stationary second scenario, (d) non-stationary second scenario. All four cases are evaluated with the same parameters (Table 2).

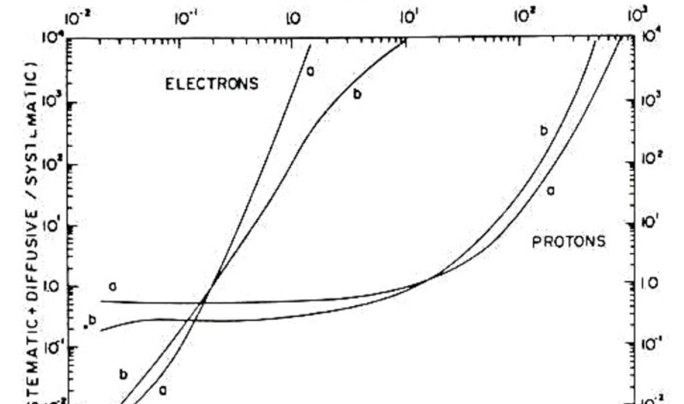

Fig. 3. The ratio R for the Fermi process in the stationary case of the first scenario. The parameters used for each curve are given in Table 1.

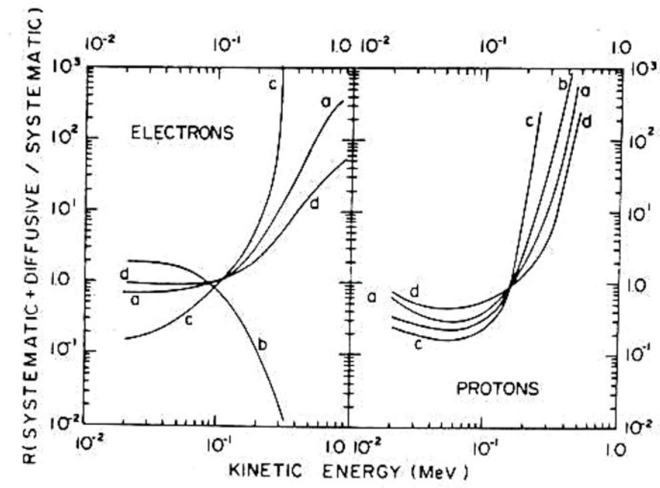

Fig. 4. The ratio R for the Fermi process in the non-stationary case of the first scenario. The parameters are given in Table 1.

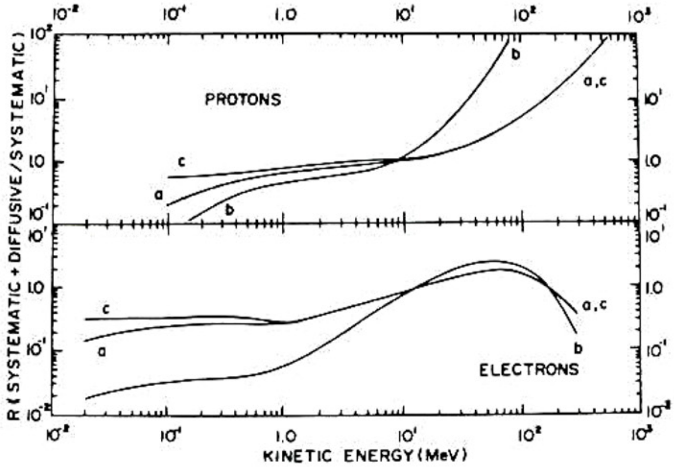

Fig. 5. The ratio R for the Fermi process in the stationary case of the second scenario. The parameters are given in Table 1.

values of the hydromagnetic and wave phase velocities, on the size scale of the inhomogeneities and magnetosonic waves, and on the magnetic field strength.

Recall from statistics that the higher the number of events the lower is the variance and the dispersion around the average value of a parameter. Here the events are interactions between particles and accelerating agents. The number of interactions is proportional to the number of "resonant" particles per energy band. Hence the importance of diffusion effects in a given energy band is related to the number of "resonant" particles, which in turn is related to the acceleration efficiency. In fact, as stated in 4.2(1), when α increases the effects of the energy spread on the spectra tend to become less important. This can be seen, for instance, in the Fermi process, where an increase in α_f is associated with an increase in turbulence which results in an increase of the local hydromagnetic velocity U and a decrease of the mean distance l between the accelerating agents ($\alpha_f \sim U^2/l$). This entails an increase in the frequency of accelerating collisions and a reduction of the statistical variance. Consequently the dispersion $<dE^2/dt>$ around the

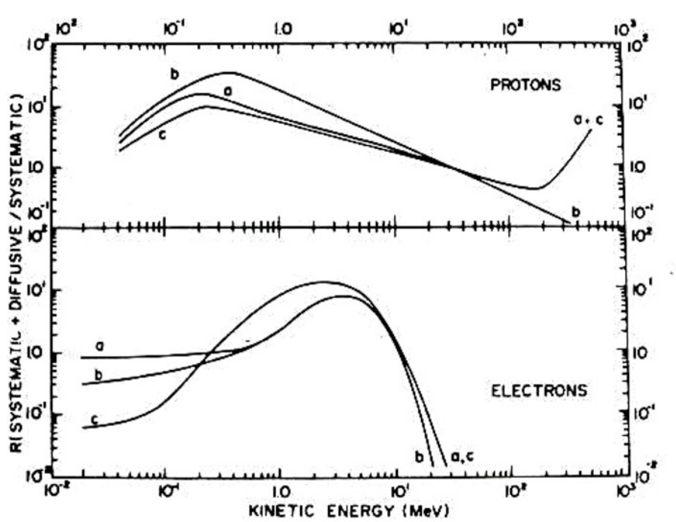

Fig. 9. The ratio R for Magnetosonic Wave Acceleration in the stationary case of the second scenario. The parameters are given in Table 2.

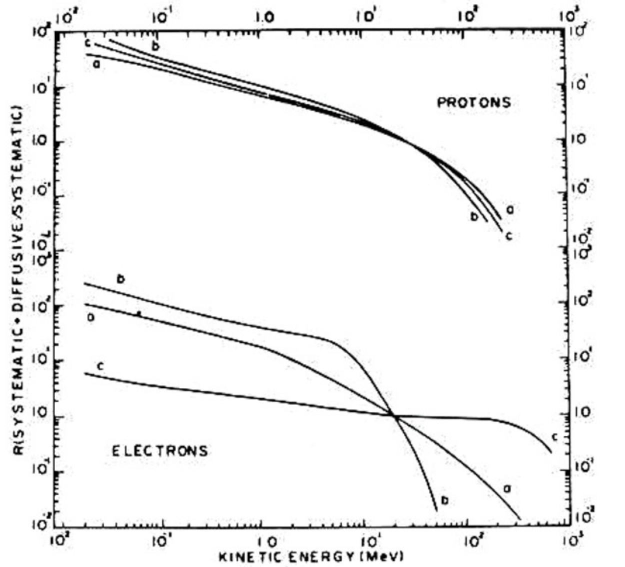

Fig. 10. The ratio R for Magnetosonic Wave Acceleration in the non-stationary case of the second scenario. The parameters are given in Table 2.

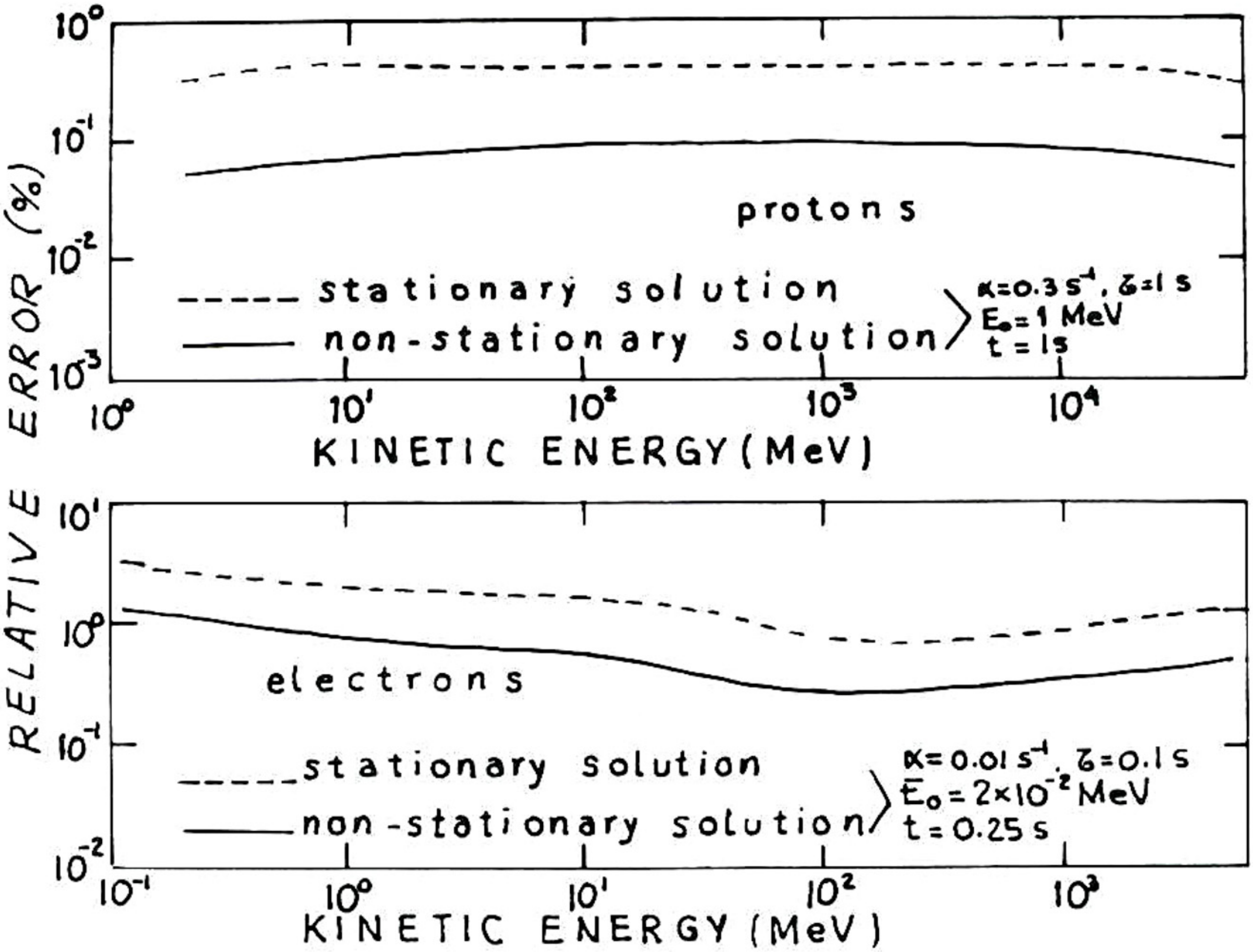

Fig. 11. Estimation of the Relative Error of the WKBJ method Sf [eq. C.3] for the time-dependent solution of eq. [32], for protons (above) and electrons (below), for typical acceleration parameters in solar particle sources.

the high suprathermal energies. Hence, during acceleration, the accelerating interactions per energy band are relatively infrequent, especially in the high energy bands. Hence, the statistical variance is high. As we saw, this implies important diffusion effects, especially in the high region of the spectrum. In contrast, particles in the second scenario are initially concentrated in a relatively narrow energetic range. Hence, during acceleration the number of accelerating interactions per energy band is relatively high and the dispersion around the average of the energy gain rate is lower than in the first scenario where the initial distribution is Maxwellian.

In both initial populations, thermal and energetic, there are more particles per energy band in the low energy section that in the high energy section of their initial spectra. Hence the effects of the energy spread during acceleration are lower in the low-energy bands than in those of higher energy. This is especially true for the first scenario where the ratio of high to low energy particles in the thermal population is lower than in the energetic distribution produced by the neutral current sheet acceleration.

Within the frame of the first scenario, if the temperature rises there is an increase of turbulence and of the acceleration efficiency parameter, i.e a higher frequency of collisions, and less diffusion from the statistical point of view. In a population of high T there are more particles above the threshold value E_o than in a population of lower T. Due to the higher number of resonant accelerating interactions, the variance around the average acceleration rate is lower for higher T, so that diffusion effects must gradually decrease as T increases as found here [4.2(3)]. Analogously, when in the second scenario the threshold value E_o is decreased, more particles in the low-energy bands are susceptible to acceleration. The frequency of accelerating collisions per energy band is increased, and the result is a minimization of energy spread in the low-energy region of the spectrum.

As for the time behavior of energy diffusion, the effects are less important in the stationary case than in the nonstationary case, [4.1(2)]. This may be understood on the basis of result 4.2(2), if in the steady-state situation the acceleration parameters have reached a value very close to the average of the systematic acceleration rate. Then, the dispersion is relatively small and the diffusion effects are less important than in the nonstationary case especially at the beginning of the acceleration. In the time-dependent case, the additional parameter (t) produces a higher dispersion of the acceleration rate at the beginning of the process, but in time this effect gradually diminishes approximating the stationary case. This can be seen from the definitions of E_o and E_i (t) given underequations (17) and (22), since for a given energy E, an increase in (t) is equivalent to a decrease in E_o or E_i. This causes more low-energy particles to participate in the acceleration process. As mentioned before, the variance in the frequency of accelerating interactions decreases, and so do the diffusion effects. Therefore, as time elapses there will be a tendency toward a stationary behavior, in which the diffusion effects will be less important. We have sometimes used very short time values to emphasize the difference between the steady-state and the time-dependent situations, as the steady state may be reached in very few seconds.

Now, from the point of view of particle dynamics, high energy particles have a larger Larmor radius, and consequently a larger interaction mean free path. Hence, for a given population it follows that there is more diffusion in the high-energy part of the spectrum than in the lower-energy part, since the collision frequency with the accelerating agents is lower. Therefore, in the Fermi process collisions in the high-energy bands are less frequent because of the limitation to interactions with $r_L \leq L \sim l$, whereas in turbulent acceleration particles may resonate even with waves much smaller than the Larmor radius ($r_L >> \lambda$). This leads to higher diffusion effects for the Fermi process at high energies, especially in the second scenario.

As for the difference in energy diffusion for different kinds of particles, obviously some difference in the importance of the R value at a given energy is expected, since the equations are not linear in mass. However, the effect is masked by the difference in particle energy between protons and electrons, which produces a sharp shift of diffusion effects along the energy axis.

Finally, it is important to point out that a quantitative analysis of particles diffusing into and out of resonant energy bands may eventually be performed by the kinematic equation of the evolution of particles in a resonant interval as derived by Canuto *et al*, (1978). This equation balances the confinement of particles in the resonant interval against their drifting out, as the particle velocity retreats from the center of maximum resonance (at V_ϕ= average wave phase velocity), taking into account the subsidiary peaks in the "diffraction resonant pattern", as emphasized by Cheng (1974). Such an analysis is beyond the scope of this work.

6. CONCLUSIONS

The relative contributions of the first and second moments of the particle distribution function have been evaluated within the context of the evolution equation in energy space of accelerated particles.

This study was done for two scenarios and two acceleration processes over the entire energy range, from thermal up to relativistic energies. We considered injection spectra with physical meaning, instead of the usual oversimplifications delta functions or power law spectra that do not contain explicit information about the physical source parameters.

The results show that diffusion in energy cannot be treated in general as a fluctuation around the average value of the systematic energy rate. It must be considered as a kind of additional energy change rate. As a matter of fact, the basic energy change process of any stochastic acceleration mechanism is the diffusion in energy, from which a certain average tendency may be isolated and designated as the systematic acceleration rate. Therefore, the so-called "fluctuational acceleration" in the Russian literature may be somewhat misleading, in the sense that diffusion is not actually translated as fluctuations of particle flux over the energy distributions. As we show, particle spectra may be modulated by an overproduction or depletion by several orders of magnitude. In other words, once we define a systematic tendency of the energy change process by means of an average tendency to increase particle energies, some situations of very high acceleration efficiency (very high turbulence density combined with a high concentration of particles with velocities around the characteristic wave phase velocity of the spectrum of turbulence) may produce negligible energy diffusion effects yet may often generate as many or much more particles than the systematic rate. This was confirmed in an earlier paper (Gallegos-Cruz and Pérez-Peraza, 1987) by using the expressions for the asymptotic case when $\beta \cong 1$. Moreover, as shown by several authors, even when the average acceleration rate is zero a well-defined particle spectrum is generated by diffusion in energy of the acceleration rate.

In calculations of the secondary flux (γ-rays, neutrons, pions), equation (6) is often solved without the second Fokker-Planck moment. It might be interesting to delimitate the situations, and if possible, the range of parameters for which the diffusion term in equation (6) can be totally neglected in solar particle generation. According to our present results this may occur mainly for scenarios with two acceleration stages, i.e., when the injected spectrum is energetic and the acceleration parameters do not change abruptly with time during acceleration so that a kind of stationary situation prevails in the source, when the acceleration parameters for protons are about $\alpha\tau \cong 0.097 \sim 0.099$, with $E_o \geq 10$ KeV, and $\alpha\tau \sim 0.093$, with $E_o \sim 1$ KeV for electrons.

However, the second Fokker-Planck coefficient cannot be ignored in solving equation (6) for a single acceleration stage from the background matter up to high energies. In such a relatively abrupt phenomenon, stationarity is not reached during the generation of the bulk of the energetic particle flux. This must be taken into account when solving the transport equation (6), for instance in deriving the flux of secondary radiation generated by the interaction of the accelerated particles with the source matter and local electromagnetic fields during acceleration. Neglecting the diffusion term in eq. (6) may lead to substantial errors.

Finally, let us mention that in particle propagation where spatial diffusion may be the main controlling factor in determining the transport process, this factor may be limited, for instance, when diffusion is constrained within magnetic flux tubes lending an anisotropic directional nature to the transport. The relative importance of spatial diffusion effects is always dependent on the scenario of particle propagation. We have also shown that diffusion in energy space is dependent on the scenario of particle acceleration, so that stochastic (diffusive) acceleration may be limited to a kind of unidirectional process by a systematic acceleration rate, given some specific values of the acceleration parameters.

APPENDIX A. STEADY-STATE SOLUTION OF THE TRANSPORT EQUATION (6).

The stationary solution of equation (6) may be derived as follows. Let us write (6) as

$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial E}(AN)-\frac{\partial^2}{\partial E^2}(DN)+\frac{N}{\tau}=Q(E,t) \qquad (A.1)$$

where $A = \langle dE/dt \rangle$, $D = \langle dE^2/dt \rangle$, $N = N(E,t)$, $Q(E,t) = q(E)\Theta(t) = q(E)$ for $t \geq 0$. In the stationary regime $(\partial N/\partial t) = 0$, so that (A.1) may be rewritten as

$$\frac{\partial^2 N}{\partial E^2}-\left[\frac{A}{D}-\frac{2}{D}\left(\frac{dD}{dE}\right)\right]\frac{dN}{dE} -\frac{1}{D}\left[\frac{1}{\tau}+\left(\frac{dA}{dE}\right)-\left(\frac{d^2D}{dE^2}\right)\right]N=-\frac{q(E)}{D} \qquad (A.2)$$

setting

$$-\left[\frac{A}{D}-\frac{2}{D}\left(\frac{dD}{dE}\right)\right]=P_1,\ -\frac{1}{D}\left[\frac{1}{\tau}+\frac{dA}{dE}-\frac{d^2D}{dE^2}\right]=-\frac{q}{D}=P_2$$

and

$$-\frac{q(E)}{D}=f(E)$$

Equation (A.2) becomes,

$$\frac{d^2N}{dE^2}+P_1\frac{dN}{dE}+P_2N=f(E) \qquad (A.3)$$

Now, defining

$$g(E)=exp\left(\int_{E_o}^{E}P_1 dE'\right) \qquad (A.4)$$

and multiplying (A.3) by g(E),

$$\frac{d}{dE}\left[g(E)\frac{dN}{dE}\right]+g(E)P_2N=g(E)f(E) \qquad (A.5)$$

Now, let us define a variable $\eta(E)$ such that $d\eta=g^{-1}(E)dE$:

$$\frac{dN}{d\eta}=g^{-1}(E)\frac{dN}{d\eta}\ and\ \frac{d}{dE}=g^{-1}(E)\frac{d}{d\eta} \qquad (A.6)$$

Introducing (A.6) in the first term of (A.5), together with (A.4), and multiplying (A.5) by $g = g(E)$, we obtain,

$$\frac{d^2N}{d\eta^2}+g^2P_2N=g^2f \qquad (A.7)$$

where $f = f(E)$. Defining $r(\eta) = g^2P_2$, and $h(\eta) = g^2f$, (A.7) becomes

$$\frac{d^2N}{d\eta^2}+r(\eta)N=h(\eta) \qquad (A.8)$$

The homogeneous part of equation (A.8) may be solved by the WKBJ method, provided the following criterion is fulfilled (e.g., Mathews and Walker, 1973):

$$\frac{1}{2}\left|\frac{dr}{d\eta}/r^{3/2}\right|<<1 \qquad (A.9)$$

In appendix C we find the conditions for which (A.9) is satisfied in solar particle sources. When the requirement (A.9) is fulfilled, the solution of the homogeneous part of (A.8), when $r < 0$ (i.e. $P_1, P_2 < 0$), is obtained by the above method as

$$N(\eta)=IN_1(\eta)+HN_2(\eta) \qquad (A.10)$$

where I and H are constants, and

$$N_{2,1}(\eta)=\left[-r(\eta)\right]^{-1/4}exp\left\{\pm\int_{\eta_o}^{\eta}\left[-r(\eta')\right]^{1/2}d\eta'\right\} \qquad (A.11)$$

In this expression N_1 corresponds to the (-) solution and N_2 to the (+) solution. We have taken into account that $r(\eta) < 0$, since $r = g^2P_2$ and $P_2= (- a/D) < 0$. The general solution of (A.8) is thus obtained by adding to (A.10) a particular solution, $N_p(\eta)$:

$$N_G=IN_1(\eta)+HN_2(\eta)+N_p(\eta) \qquad (A.12)$$

$N_p(\eta)$ may be derived by the method of variation of Lagrange parameters (which in this case is equivalent to the Green function method), leading to the following result,

$$N_p(\eta)=-N_1(\eta)\int_{n_o}^{n}\frac{N_2(\eta')h(\eta')d\eta'}{W(N_1,N_2)}+N_2(\eta)\int_{n_o}^{n}\frac{N_1(\eta')h(\eta')d\eta'}{W(N_1,N_2)} \qquad (A.13)$$

where $W(N_1, N_2) = N_1(dN_2/d\eta) - N_2(dN_1/d\eta)$ is the Wronskian of N_1 and N_2.To evaluate this Wronskian, let us put for simplicity $j(\eta) = - r(\eta) = - g^2P_2$. Now, deriving (A.11) with respect to η, we obtain

$$\frac{dN_1}{d\eta}=j^{1/4}exp\left(-\int_{\eta_o}^{\eta}j^{1/2}d\eta'\right)-\frac{1}{4}j^{-5/4}exp\left(-\int_{\eta_o}^{\eta}j^{1/2}d\eta'\right)j'$$

$$\frac{dN_2}{d\eta}=j^{1/4}exp\left(\int_{\eta_o}^{\eta}j^{1/2}d\eta'\right)-\frac{1}{4}j^{-5/4}exp\left(\int_{\eta_o}^{\eta}j^{1/2}d\eta'\right)j'$$

where $j'= dj/d\eta$, in such a way that $W(N_1,N_2) = 2$, and therefore (A.12) may be rewritten as

$$N_G(\eta)=IN_1(\eta)+HN_2(\eta)-\frac{N_1(\eta)}{2}\int_{\eta_o}^{\eta}N_2(\eta')h(\eta')d\eta'+\frac{N_2(\eta)}{2}\int_{\eta_o}^{\eta}N_1(\eta')h(\eta')d\eta' \qquad (A.14)$$

Since the energy spectrum of cosmic particles is a decreasing function of energy, and $\eta \cong E$ [according to (A.6)], we may cancel the 2nd and 4th term of (A.14) because N_2 is an increasing function of energy. Using $h = g^2f = - g^2q/D$, (A.14) may be written as

$$N_G(\eta)=IN_1(\eta)-\frac{1}{2}N_1(\eta)\int_{\eta_o}^{\eta}N_2(\eta')g^2(\eta')q(\eta')d\eta'/D \qquad (A.15)$$

Now, introducing the definitions of $r(\eta)$, g and η, equation (A.11) may be rewritten as,

$$N_{2,1}(E)=a^{-1/4}(E)D^{1/4}(E)\,exp\left[-\int_{E_o}^{E}P_1dE'\pm\int_{E_o}^{E}P_2^{1/2}dE'\right] \qquad (A.11a)$$

using (A.11a) in (A.15) and recovering the initial variable (E), (A.15) becomes,

$$N(E)=\frac{ID^{1/4}(E)}{D^{1/4}(E_o)}exp\left[-\frac{1}{2}\int_{E_o}^{E}P_1dE'-\int_{E_o}^{E}P_2^{1/2}dE'\right]+\frac{D^{1/4}(E)}{2a^{1/2}}\int_{E_o}^{E}\frac{q(E')}{D^{3/4}(E')}+exp\left[-\frac{1}{2}\int_{E'}^{E}P_1dE''-\int_{E'}^{E}P_2^{1/2}dE''\right]dE' \qquad (A.16)$$

where E' and E" are integration variables in E. The evaluation of I in (A.16) requires boundary conditions which depend on the assumed scenario. For the first scenario in this paper, the low energy extreme corresponds to the local thermal population, so that I is obtained from the condition that when particle energy $E \rightarrow$ thermal energies, the first term of (A.16) approaches the Maxwell thermal distribution evaluated at the characteristic threshold value (E_o) for injection into the acceleration process. Similarly, in the second scenario I tends toward $q(E_o)$ as E tends to E_o.

The explicit evaluation of (A.16) for each acceleration mechanism requires solving the corresponding integrals:

(a) for the Fermi mechanism, from eqs.(8) and (10) $A(E) = (4/3)\alpha\beta W$, and $D(E) = (1/3)\alpha\beta^3W^2$, with $W = E + mc^2$ and $\beta = (W^2- m^2c^4)^{1/2}/W$, so that,

$$-\frac{1}{2}\int_{E_o}^{E}P_1dE'=ln\left(\frac{\beta_o}{\beta}\right) \text{ and }$$

$$-\int_{E'}^{E}P_2^{1/2}dE''=-\left(\frac{3a}{\alpha}\right)^{1/2}J(E',E) \qquad (A.17)$$

where, $J(E,E') = \tan^{-1}(\beta^{1/2}) - \tan^{-1}(\beta'^{1/2})$

$$+(1/2)ln\left[\frac{(1+\beta^{1/2})(1-\beta'^{1/2})}{(1-\beta^{1/2})(1+\beta'^{1/2})}\right]$$

(b) In the case of turbulent acceleration by magnetosonic waves, according to eqs. (12) and (13), $A(E) = \alpha_m\beta^2 W$ and $D(E) = (\alpha_m/2)\beta^4 W^2$, where α_m was defined in eq. (14). The corresponding integrals in this case are

$$-\frac{1}{2}\int_{E_o}^{E} P_1 dE' = ln\left[\left(\frac{\beta_o}{\beta}\right)^{15/4}\left(\frac{W_o}{W}\right)^{7/4}\right]$$

$$-\int_{E_o}^{E} P_2^{1/2} dE' = ln\left[\left(\frac{\beta W}{\beta' W'}\right)\right]^{-\left(\frac{a}{2\alpha_m}\right)^{1/2}} \qquad (A.18)$$

APPENDIX B. TIME DEPENDENT SOLUTION OF THE TRANSPORT EQUATION (6)

The non-stationary solution of equation (6) may be obtained by the following procedure. Let us take the Laplace transform of (6),

$$S\tilde{N}(E,S) - N(E,S) + \frac{d}{dE}\left[A\tilde{N}(E,S)\right] - \frac{d^2}{dE^2}\left[D\tilde{N}(E,S)\right] + \frac{\tilde{N}(E,S)}{\tau} = \tilde{Q}(E,S) \qquad (B.1)$$

where S is the Laplace variable. Developing the 3rd and 4th terms of (B.1) and reordering them,

$$\frac{d^2\tilde{N}}{dE^2} - \left[\frac{A}{D} - \frac{2}{D}\left(\frac{dD}{dE}\right)\right]\frac{d\tilde{N}}{dE} - \left[\frac{S+a}{D} + \frac{1}{D}\frac{dA}{dE} - \frac{1}{D}\left(\frac{d^2D}{dE^2}\right)\right]\tilde{N} = -\frac{\tilde{q}}{D} - \frac{N(E)}{D} \qquad (B.2)$$

where $\tilde{N} = \tilde{N}(E,S)$, $\tilde{q} = \tilde{Q}(E,S)$ and $N(E) = N(E,0)$. By introducing P_1, P_2 and $f(E)$ in (B.2), we obtain

$$\frac{D^2\tilde{N}}{dE^2} + P_1\frac{d\tilde{N}}{dE} + P_2\tilde{N} = f(E) \qquad (B.3)$$

which is identical to (A.3). Its specific solution in analogy to (A.15) may be written as,

$$\tilde{N}_G(\eta,S) = I\tilde{N}_1(\eta,S) - \frac{1}{2}\tilde{N}_1(\eta,s)\int_{\eta_o}^{\eta}\tilde{N}_2(\eta',s)h(\eta',s)dn' \qquad (B.4)$$

where $h = g^2 f(E)$, g and η were defined in (A.8), (A.4) and (A.6). Now, using (A.11) in (B.4) with $r(\eta) = g^2 P_2$, we obtain

$$\tilde{N}_G(E,S) = \frac{I}{P_2^{1/4}} exp\left[-\frac{1}{2}\int_{E_o}^{E} P_1 dE' - \int_{E_o}^{E} P_2^{1/2} dE'\right] + \frac{1}{2P_2^{1/4}} exp\left[-\frac{1}{2}\int_{E_o}^{E} P_1 dE'\right] x \int_{E_o}^{E} \frac{\tilde{Q}(E',S)+\tilde{N}(E',O)}{D(E')P_2^{1/4}(E')}$$

$$exp\left[-\frac{1}{2}\int_{E_o}^{E'} P_1 dE'' - \frac{1}{2}\int_{E'}^{E} P_2^{1/2} dE''\right]dE' \qquad (B.5)$$

where I is a constant to be evaluated from the boundary conditions (discussed below eq. B.8); with the consideration of a steady-state injection spectrum $Q(E,t) = q(E)\Theta(t) = q(E)$, hence, $Q(E,S) = q(E)/S$, so that (B.5) becomes,

$$\tilde{N}(E,S) = \frac{ID^{1/4}(E)}{[S+a(E)]^{1/4}} x$$

$$exp\left[-\frac{1}{2}\int_{E_o}^{E} P_1 dE' - \int_{E_o}^{E} \frac{[S+a(E')]^{1/2} dE'}{D^{1/2}(E')}\right] + \frac{D^{1/4}(E)}{2[S+a(E)]^{1/4}} exp\left[-\frac{1}{2}\int_{E_o}^{E} P_1 dE'\right]$$

$$x\int_{E_o}^{E} \frac{[q(E')/S + N(E',O)]}{D^{3/4}(E')[S+a(E')]^{1/4}}$$

$$exp\left[\frac{3}{2}\int_{E_o}^{E'} P_1 dE'' - \int_{E'}^{E} \frac{[S+a(E'')]^{1/2}}{D^{1/2}(E'')} dE''\right]dE' \qquad (B.6)$$

where a was defined below (A.2) in Appendix A. In solar particles sources the main contribution to a is determined by $1/\tau$, thus we set $a(E) = \tau^{-1} + \overline{F}$, as in connection with the spectrum (27). $\overline{F}$ is evaluated by taking the average between the injection energy E_o and the corresponding value of E so that (B.6) may be written as,

$$N(E,t) = ID^{1/4}(E) exp\left[-I_1(E_o,E)\right]$$

$$L^{-1}\left\{(a+S)^{-1/4} exp\left[-(S+a)^{1/2} I_2(E_o,E)\right]\right\} + \frac{D^{1/4}(E)}{2}$$

$$exp\left[I_1(E_o,E)\right]\int_{E_o}^{E} \frac{q(E')}{D^{3/4}(E')} exp\left[I_1(E_o,E')\right]$$

$$L^{-1}\left\{\frac{exp\left[-(S+a)^{1/2} I_2(E',E)\right]}{S(S+a)^{1/2}}\right\}dE' + \frac{D^{1/4}(E)}{2}$$

$$exp\left[I_1(E_o,E)\right]\int_{E_o}^{E} \frac{N(E',0)}{D^{3/4}(E')} exp\left[I_1(E_o,E')\right]$$

$$L^{-1}\left\{\frac{exp\left[-(S+a)^{1/2} I_2(E',E)\right]}{(S+a)^{1/2}}\right\}dE' \qquad (B.7)$$

where $I_1(E_o,E) = \left(\frac{1}{2}\right)\int_{E_o}^{E} P_1 dE'$,

$I_2(E',E) = \int_{E'}^{E} P_2^{1/2} dE''$ and L^1

stands for the Laplace inverse transform. Obviously, since I_1 and I_2 depends on P_1 and P_2, they need to be evaluated for each acceleration process. The Laplace inverse transforms to be used are

$$f(s-b) \rightarrow F(t)exp(bt), \qquad b = constant$$
$$exp(-bs^{1/2})/s^{1/2} \rightarrow exp(-b^2/4t)/(\pi t)^{1/2}$$
$$exp(-bs^{1/2}) \rightarrow b.exp(-b^2/4t)/4\pi t^3$$
$$s^{-1/4} \rightarrow t^{-3/4}/\Gamma(1/4) \qquad \Gamma = \text{gamma function}$$

$$L^{-1}[f(S)g(S)] = \int_o^t F(t-\mu)G(\mu)d\mu \quad \text{(convolution theorem)}$$

Thus we obtain,

$$L^{-1}\left\{\frac{exp\left[-(S+a)^{1/2}I_2\right]}{S(S+a)^{1/2}}\right\} = \int_o^t (\pi t')^{-1/2} exp\left[-at'-I_2^2(E',E)/4t'\right]dt'$$

$$L^{-1}\left\{\frac{exp\left[-(S+a)^{1/2}I_2\right]}{(S+a)^{1/4}}\right\} = \frac{1}{2\pi^{1/2}r(1/4)}\int_o^t \frac{exp\left[-I_2^2/(4t')\right]}{t'^{3/2}(t-t')^{3/4}}dt'$$

$$L^{-1}\left\{\frac{exp\left[-(S+a)^{1/2}I_2\right]}{(S+a)^{1/2}}\right\} = (\pi t)^{-1/2} exp\left[-at - I^2(E',E)/(4t')\right]$$

With these results equation (B.7) may be rewritten as,

$$N(E,t) = \frac{ID^{1/4}(E)}{2\pi^{1/2}r^{1/4}} I_2(E_o,E)exp\left[-at - I_l(E_o,E)\right]$$
$$\int_o^t \frac{exp\left[-I_2^2(E_o,E)/(4t')\right]}{t'^{3/2}(t-t')^{3/4}}dt' + \frac{D^{1/4}(E)}{(4\pi t)^{1/2}} exp\left[-at - I_l(E_o,E)\right]$$
$$\int_{Eo}^E \frac{N(E',O)}{D^{3/4}(E')} exp\left[I_l(E_o,E') - I_2^2(E',E)/(4t)\right]$$
$$+ \frac{D^{1/4}(E)}{(4\pi)^{1/2}} exp\left[-I_l(E_o,E)\right]\int_o^t \frac{dt'}{t'^{1/2}}$$
$$\int_{Eo}^E \frac{q(E')}{D^{3/4}(E')} exp\left[-at' - I_l(E_o,E') - \frac{I_2^2(E',E)}{(4t')}\right]d \tag{B.8}$$

where $I_l(E',E) = \left(\frac{1}{2}\right)\int_{E'}^E P_l dE''$, and $I_2^2(E',E)$ $=[I_2(E',E)]^2$. Formally, equation (B.8) is the general solution of equation (6); however, within the scope of energetic particle evolution during their acceleration, the 2nd and 3rd terms of (B.8) must satisfy all boundary conditions at the low and high extremes of the energy spectrum. Thus the first term becomes unnecessary, which implies in this case that I = 0, so that our general solution within the scope of energetic particle generation in cosmic sources reduces to equation (32) of the text.

APPENDIX C. APPLYING CONDITIONS OF THE WKBJ METHOD

The general requirement for applying the WKBJ method to the solution of the homogeneous differential equation (A.2) was given in (A.9) as

$$S = \frac{1}{2}\left|\frac{dr}{d\eta}/r^{3/2}\right| << 1 \tag{C.1}$$

where according to the definitions given in Appendix A, we have,

$$r^{3/2}(E) = -\left(\frac{a}{D}\right)^{3/2} exp\left[3\int_{Eo}^E P_l dE'\right]$$

$$\frac{dr}{d\eta} = \frac{dr}{dE}\frac{dE}{d\eta} = aD^{-2}\left[3\left(\frac{dD}{dE}\right) - 2A\right]exp\left[3\int_{Eo}^E P_l dE'\right]$$

so that equation (C.1) may be rewritten as

$$S = \left|3\frac{dD}{dE} - 2A\right|/\left(4aD^{1/2}\right) \tag{C.2}$$

Now let us apply (C.2) to the two specific acceleration processes studied in this work.

(a) For the Fermi mechanism, we have,

$$S_f = 0.29\left[\alpha_f/\bar{a}(E_o,E)\right]^{1/2}\left|1-3\beta^2\right|/\beta^{1/2} \tag{C.3}$$

Since $a(E) = \tau^{-1} + F(E)$, with $F(E) = (\alpha/3)(\beta^{-1}+3\beta - 2\beta^3)$, we may define $a(E_o,E) = \tau^{-1} + (1/2)[F(E_o) + F(E)]$, with $F(E_o)$ defined as F(E) in terms of β_o. For the range of parameters used in this paper, $\alpha = (0.04 - 0.75)s_{-1}$, i.e., $\alpha\tau \cong 0.004 \sim 0.4$ for protons, and $\alpha = (0.01 - 0.2)s^{-1}$, i.e., $\alpha\tau \cong 0.005 \sim 0.1$ for electrons, (C.1) is fulfilled. Let us consider a range of particle velocities between thermal energies ($\beta_o = 0.5$) and ultrarelativistic energies ($\beta \sim 1$). Then equation (C.3) shows us that in all cases $S_f \leq 0.1$.

(b) For turbulent acceleration by magnetosonic waves (C.2) is transformed into

$$S_m = 0.7\left[\alpha_m/\bar{a}(E_o,E)\right]^{1/2}\left|1\ 1-6\beta^2\right| \tag{C.4}$$

where $a(E_o,E) = \tau^{-1} + (1/2)[G(E\ ) + G(E_o)]$, $G(E) = \alpha_m(23\beta^2 - 12\beta^4 - 14)$ and $G(E_o)$ is defined in terms of β_o in the same form as G(E). Again, for the range of parameters used in this paper, it can easily be verified that $S_m \leq 0.1$.

Therefore, the analysis of (C.2) and (C.4) shows that the conditions for which S_f or $S_m << 1$ are fulfilled when $\alpha/a << 1$.

BIBLIOGRAPHY

ACHTERBERG, A., 1979. *Astronom. and Astrophys. 76*, 276.

ACHTERBERG, A., 1981. *Astronom. and Astrophys. 97*, 259.

BARNES, A.,1966. *Phys. Fluids 9*, 1483.

BERGER, J M., W.A. NEWCOMB, J.M. DAWSON, E.A. FRIEDMAN and A. LENARD, 1958. *Phys. Fluids 1*, 301.

CANUTO, V., C. K. CHOU and L. FASSIO-CANUTO, 1978. *Fund. Cosmic Phys. 3*, 221.

CHENG, F.E., 1974. Introduction to Plasma Physics, Plenum Press.

DAVIS, L., 1956. *Phys. Rev. 101*, 351.

EICHLER, D., 1979. *Ap. J. 229*, 413.

EILEK, J.A., 1979. *Ap. J. 230*, 373.

EILEK, J.A., 1984. *Ap. J. 277*, 820.

FERMI, E., 1949. *Phys. Rev. 75*, 1169.

FISK, L.A., 1976. *J. Geophys. Res., 81*, 4633.

FORMAN, M.A. and G. M. WEBB, 1985. *In:* Collisionless shocks in the Heliosphere, Geophysical Monographs, ed. by Stone R.G. and Tsurutani, B.T., 34, 91,

GALLEGOS-CRUZ, A. and J. PEREZ-PERAZA, 1987, *Rev. Mex. Astronom. y Astrofís, 14*, 700.

GINZBURG, V. L.,1958. *In:* Progress in Elementary Particle and Cosmic Ray Physics, Interscience, New York 4, 339.

GINZBURG, V. L. and L. I. SYROVATSKII, 1964. The Origin of Cosmic Rays, Pergamon Press.

JONES, F. C.,1991. *In:* Particle Acceleration in Cosmic Plasmas, AIP Conference Proceedings, 264, 71.

KAPLAN, S. A. and V. N. TYSTOVICH, 1973. Plasma Astrophysics, Pergamon Press (Oxford).

KULSRUD, R. M. and A. FERRARI, 1971. *Astrophys. and Space Sci. 12*, 302.

LACOMBE, C. and A. MANGANEY, 1969. *Astronom. and Astrophys. 1*, 325.

MATHEWS, J. and R. L. WALKER, 1973. *In:* Mathematical Methods of Physics, Addison-Wesley Publ. Co., p. 27.

MELROSE, D. B.,1975. *Solar Phys. 43*, 79.

MELROSE, D. B. and J. C. BROWN, 1976. *Mont. Not. R. Astr. Soc. 176*, 15.

MELROSE, D. B., 1980. Plasma Astrophysics, Vol. II, Gordon & Breach .

MELROSE, D.B., 1993. Preprint, 142th IAU Colloquium on Particle Acceleration in Astrophysical Plasma and Particle Acceleration in Cosmic Plasma, AIP Conference Proceedings 264, 3.

MILLER, A. J., N. GUESSOUM and R. RAMATY, 1990. *Ap. J. 361*, 701.

MULLAN, D. J., 1980. *Ap. J. 237*, 244.

PARKER, E. N. and D. A. TIDMAN, 1958. *Phys. Rev. 11*, 1206.

PEREZ-PERAZA, J., 1975. *J.Geophys. Res. 80*, 3535.

PEREZ-PERAZA, J., M. GALVEZ and R. LARA, 1977. *Space Res. 18*, 365, 1978 and Proc. 15th ICRC, 5, 23.

PEREZ-PERAZA, J., 1981. Comptes Rendues Acad. Sci. Paris 292, 725, 1981 and Proc. 17th ICRC, 3, 55.

PEREZ-PERAZA, J., J. MARTINELL and A. VILLAREAL, 1982. *Adv. Space Res. 2*, 197.

PEREZ-PERAZA, J. and A. GALLEGOS-CRUZ, 1994, *Ap. J. Supl. Series 90-2 (in press)*.

PRIEST, E., 1973. *Ap. J. 181*, 227.

SCHATTEN, K. H. and D. J. MULLAN, 1977. *J. Geophys. Res., 82*, 5609.

SCHATZMAN, E., 1966. *In:* Summer School of Theoretical Physics, Les Houches, Gordon & Breach Eds., p. 229.

SWANN, W. F. G., 1933. *Phys. Rev. 43*, 217.

TVERSKOI, B. A., 1968. *Soviet Phys. JEPT 25*, 317, 1967, or in the English version, 26, 821.

TYSTOVICH, V. N., 1977. Theory of Turbulent Plasma, Ed. by Plenum Press, New York.

UNSOLD, A., 1951. *Phys. Rev. 82*, 857.

J. Pérez-Peraza[1] and A. Gallegos-Cruz[2]

[1] *Instituto de Geofísica, UNAM, Cd. Universitaria 04510 México, D.F., México*

[2] *I. N. A. O. E., A. P. 51, 72000 Tonantzintla, Puebla, México*

[2]Currently studying at UAM, Iztapalapa, México D.F., MEXICO